Mathematical Reviews

Edited by

O. Neugebauer **J. D. Tamarkin** **O. Veblen**

W. Feller, *Executive Editor*

Vol. 6, No. 1 **January, 1945** **pp. 1–28**

TABLE OF CONTENTS

THE MICROFILM SERVICE

Through the services of the Brown University Photographic Laboratory one can obtain, either on microfilm or as photoprint, the complete text of any article listed in Mathematical Reviews or available in the Brown University Library, except books and material of which the reproduction is prohibited by copyright laws.

Procedure for Ordering Microfilm or Photoprint

1. Give your name and address.

2. State whether you wish microfilm or photoprint.

3. (a) If the article in question has been listed in Mathematical Reviews, give the MF number printed at the end of the title. Titles without MF numbers are copyrighted.

   (b) If the article has not been reviewed (for instance because published prior to 1939) give as accurately as possible the author, title, journal, volume, year and pages.

4. Enclose cash for the amount of the order. U. S. currency or check, made payable to Brown University, will be accepted. The minimum charge is 45¢.

5. Mail to: *Brown University Photographic Laboratory, Providence, Rhode Island.*

Cost

MICROFILMS are furnished for

- 15¢ for articles with less than 25 pages.
- 30¢ for articles with more than 24 pages, but less than 50 pages.
- 50¢ for articles with more than 49 pages, but less than 100 pages.
- $1.00 for articles with more than 99 pages.

PHOTOPRINTS cost 20¢ per sheet, each sheet containing two pages of text, except in the case of unusually large journals (larger than 7" by 10"). In the latter case you will be informed of the extra charge before the order is filled, unless you have indicated on your order that the reprint is desired even at the higher price of 20¢ per page.

MINIMUM CHARGE. If an order totals less than 45¢ the cash included should be raised to 45¢ in order to cover the basic expenses of this service.

Microfilm and photoprints with MF numbers show black writing on white background. Articles which are not reviewed, and which therefore do not have an MF number, are reproduced only in negative films or prints, i.e. white letters on black background. Microfilms can be read in any reading machine adapted to standard 35 mm. double perforated film. Photoprints are on sheets 10" by 14" (about 25 by 35 cm.).

ONE-SIDE EDITION OF MATHEMATICAL REVIEWS

An edition of Mathematical Reviews printed on only one side of the paper is available to persons interested in making card files of reviews or wishing to add remarks to reviews in the future. This special edition may be obtained for an additional payment of $1.00. A regular current subscription can be changed to a one-side subscription by paying the additional $1.00. This edition is folded but not stitched.

MATHEMATICAL REVIEWS

Published monthly, except August, by

The American Mathematical Society, Prince and Lemon Streets, Lancaster, Pennsylvania

Sponsored by

The American Mathematical Society
The Mathematical Association of America
The Edinburgh Mathematical Society
Academia Nacional de Ciencias Exactas, Fisicas y Naturales de Lima
Het Wiskundig Genootschap te Amsterdam
The London Mathematical Society
Union Matematica Argentina

Editorial Office

Mathematical Reviews, Brown University, Providence, R. I.

Subscriptions: Price $13 per year ($6.50 per year to members of sponsoring societies). Checks should be made payable to Mathematical Reviews. Subscriptions should be addressed to Mathematical Reviews, Lancaster, Pennsylvania, or Brown University, Providence, Rhode Island.

This publication was made possible in part by funds granted by the Carnegie Corporation of New York, the Rockefeller Foundation, and the American Philosophical Society held at Philadelphia for Promoting Useful Knowledge. These organizations are not, however, the authors, owners, publishers, or proprietors of this publication, and are not to be understood as approving by virtue of their grants any of the statements made or views expressed therein.

Entered as second-class matter February 3, 1940 at the post office at Lancaster, Pennsylvania, under the act of March 3, 1879. Accepted for mailing at special rate of postage provided for in the Act of February 28, 1925, embodied in paragraph 4, section 538, P. L. and R. authorized November 9, 1940.

*Ko
sch
194
Th
Anwe
lisher

von S
run
An
Le
that
$pq=$
a $\delta>$
cosin
p_2q_2
cos

Two
comm
the f
and

wher
a con
No
whic
the t
ditio
a giv
wher
lowe
corre
the a
cont
nary
of Z

Wey
the
eq
Ac
Th
wher
neigh
F an
appr
teris
are
$0<l$

Mathematical Reviews

Vol. 6, No. 1 JANUARY, 1945 Pages 1-28

ANALYSIS

Differential Equations

★Kowalewski, Gerhard. Integrationsmethoden der Lieschen Theorie. J. W. Edwards, Ann Arbor, Michigan, 1944. viii+221 pp. $4.65.

The original appeared as vol. 15 of Mathematik und ihre Anwendungen in Monographien und Lehrbüchern; publisher: Akademische Verlagsgesellschaft, Leipzig, 1933.

von Schwarz, Josefa. Eine Methode zur Verallgemeinerung der gewöhnlichen Differentialgleichungen. Math. Ann. 118, 497–517 (1942). [MF 10712]

Let R be a space with a uniformly continuous distance; that is, any two points p, q of R have a distance pq with $pq=qp>0$, $p\neq q$, and $pp=0$, and for every $\epsilon>0$ there is a $\delta>0$ such that $pp'+qq'<\delta$ implies $|pq'-p'q|<\epsilon$. The cosine of the angle between two directed segments p_1q_1 and p_2q_2 in a Euclidean space is

$$\cos\,(p_1, q_1; p_2, q_2) = [p_1q_2^2+q_1p_2^2-p_1p_2^2-q_1q_2^2]/[2p_1q_1^2\cdot p_2q_2^2].$$

Two oriented arcs C_1 and C_2 in R are said to touch at the common point r if for a given $\epsilon>0$ there exists a $\delta>0$ with the following property: when p_i precedes q_i on C_i, $i=1, 2$, and $p_ir<\delta$, $q_ir<\delta$, then

$$1-\epsilon\leqq\cos\,(p_1, q_1; p_2, q_2)\leqq 1+\epsilon,$$

where cos is defined by the above equation. An arc C has a continuous tangent at p if it touches itself at p.

Now let every point x of R be the origin of an arc C_x which has everywhere a continuous tangent, and such that the tangent of C_x at x depends continuously on x (this condition is expressed in terms of the cosine function). Then a given point r is the origin of an arc C which has everywhere a continuous tangent, finite upper length and positive lower length, and which touches at each of its points x the corresponding arc C_x. Actually, more general functions than the above cosine are admitted. It is shown that this theorem contains the known existence theorem (of Peano) on ordinary differential equations and also a more general theorem of Zaremba [Bull. Sci. Math. (2) 60, 139–160 (1936)].
H. Busemann (Chicago, Ill.).

Weyl, Hermann. Concerning a classical problem in the the theory of singular points of ordinary differential equations. Revista Ci., Lima 46, 73–112 (1944) = Actas Acad. Ci. Lima 7, 21–60 (1944). [MF 10761]

The author considers the equations $\dot{x}=F(x, y)$, $\dot{y}=G(x,y)$, where differentiation is with respect to a parameter t, in the neighborhood of the origin, where both F and G vanish; F and G are both assumed to be linear, to an ever improving approximation, as (x, y) approaches the origin. The characteristic values associated with the linear parts of F and G are denoted by k and l. The author considers the cases $0<l<k$ and $l<0<k$. Under an affine transformation we may replace $F(x, y)$ by $kx+f(x, y)$ and similarly G by $-ly+g(x, y)$.

The author's hypothesis concerns the nature of f and g and is a type of Lipschitz condition. His condition is certainly satisfied, for instance, in the case where the first order partial derivatives of f and g exist and are continuous near the origin and are less in magnitude than Cr^δ, where C and δ are positive constants and $r=\max(|x|, |y|)$. Under such a comparatively weak hypothesis, the author obtains a number of remarkably precise quantitative theorems concerning the solutions of the equation near the origin.

Given any number a, the author proves the existence of a unique solution $X(t)$, $Y(t)$ satisfying $e^{kt}X(t)\to a$, $e^{kt}Y(t)\to 0$ as $t\to+\infty$. Aside from translations in t, it suffices to take $a=\pm 1$. Thus there are two and only two solutions which approach the origin from opposite sides along the x axis. Among the author's results are theorems giving the quantitative relationship between any solution near the origin and these exceptional solutions.
N. Levinson.

Shohat, J. A new analytical method for solving van der Pol's and certain related types of non-linear differential equations, homogeneous and non-homogeneous. J. Appl. Phys. 14, 40–48 (1943).

Shohat, J. On van der Pol's and non-linear differential equations. J. Appl. Phys. 15, 568–574 (1944).

The first of the above papers contains no new results and contains seriously erroneous statements. However, in the second of the above papers the author more than makes up for the failings of his first paper.

He considers the nonlinear differential equation $\ddot{u}-\epsilon F(u)\dot{u}+u=0$, $\epsilon\to 0$, where $F(u)$ is of such a nature that the equation has a periodic solution. As $\epsilon\to 0$, the character of the periodic solution is known to be of sinusoidal form. By an application of the Parseval theorem the author is not only able to show this but also obtains strong bounds for the higher harmonics in the Fourier series representing the solution. Using the case $F(u)=1-u^2$, the van der Pol equation, as an illustration, the author obtains a very interesting series for the periodic solution of the differential equation which yields results for large values of ϵ as well as for small values.
N. Levinson (Cambridge, Mass.).

Titchmarsh, E. C. An extension of the Sturm-Liouville expansion. Quart. J. Math., Oxford Ser. 15, 40–48 (1944). [MF 10688]

The author remarks that the equation $(L-w)f=0$, in which

$$f=\begin{pmatrix} f_1(x) \\ f_2(x) \end{pmatrix},\quad L=\begin{bmatrix} p(x) & g(x)+\dfrac{d}{dx} \\ g(x)-\dfrac{d}{dx} & r(x) \end{bmatrix},$$

gives rise to a formula for the expansion of an arbitrary function in characteristic solutions, and proceeds with the details. This appears to be a mere special case of well-known and much more general theory relative to differential systems

$$y'(x)=\{\lambda A(x)+B(x)\}y(x),$$

in which $y(x)$ is a vector and the $A(x)$, $B(x)$ are square matrices. *R. E. Langer* (Madison, Wis.).

Makai, E. Über die Nullstellen von Funktionen, die Lösungen Sturm-Liouville'scher Differentialgleichungen sind. Comment. Math. Helv. 16, 153–199 (1944).

The primary purpose of this paper is the derivation of upper and lower bounds for the roots of the Hermite and Laguerre polynomials. The approach to the problem is taken through the differential equations, and two methods are prominent. These are: (i) the deduction and application of some general theorems for the appraisal of the characteristic values and the intervals between zeros of the characteristic functions of the boundary problem

$$(d/dx)p(x)(d/dx)y+\lambda g(x)=0,\quad y(a)=y(b)=0,$$

in which $p(x)$ and $g(x)$ are positive and of class C'' on the interval $a<x<b$, and in which a more involved expression depending upon $p(x)$ and $g(x)$ maintains its sign; and (ii) the transformation of the differential equation into a form to which the methods of Sturm are readily applicable. With the roots of the Hermite polynomial $H_n(x)$ designated in the order of descending values by $x_{n,1}, x_{n,2}, \cdots, x_{n,n}$, it is shown that, for $k\leqq[n/2]$,

$$\int_{x_{n,k}}^{(2n+1)^{\frac{1}{2}}}\{2n+1-x^2\}^{\frac{1}{2}}dx=(k-\tfrac{1}{4}+\epsilon_{n,k})\pi,$$

and numerical bounds for the $\epsilon_{n,k}$ are given. Bounds directly applicable to the $x_{n,k}$ are also derived. In the case of the Laguerre polynomials $L_n^{(\alpha)}(x)$, whose roots in the order of increasing values are designated by $x_{n,1}^{(\alpha)}, x_{n,2}^{(\alpha)}, \cdots, x_{n,n}^{(\alpha)}$, bounds are obtained for $x_{n,n}^{(\alpha)}$ and for

$$\int_0^{x_{n,k}^{(\alpha)}}((4n+2\alpha+2-x)/4x)^{\frac{1}{2}}dx,\qquad |\alpha|\leqq\tfrac{1}{2},$$

or

$$\int_0^{(x_{n,k}^{(\alpha)})^{\frac{1}{2}}}\Re(4n+2\alpha+2-x^2+(\tfrac{1}{4}-\alpha^2)/x^2)^{\frac{1}{2}}dx,\quad \alpha\geqq\tfrac{1}{2}.$$

The roots of the Legendre polynomials, the Bessel functions of order zero and the Mathieu functions are more briefly discussed. *R. E. Langer* (Madison, Wis.).

Laguardia, Rafael and Levi, Beppo. On the representation by integrals of some functions defined by Taylor expansions and its application to the solution of partial differential equations. Publ. Inst. Mat. Univ. Nac. Litoral 4, 205–232 (1943). (Spanish. English summary) [MF 10835]

The authors consider several types of differential equations, and systems of equations mainly with constant coefficients of the general type

$$\partial^m z_i/\partial x^m-\sum_{p,j}\partial^p z_j/\partial y_1\partial y_2\cdots=f_i(x,y_1,y_2,\cdots).$$

The initial values of the functions and their derivatives are taken to be 0. The corresponding solutions are represented by real and complex integrals. The simplest examples of the equations and their solutions are

(I) $$\partial^m z/\partial x^m-a(y_1,\cdots,y_k)z=f(x,y_1,\cdots,y_k),$$

$$z=\alpha^{1-m}\int_0^x\sigma_m(\alpha x-\alpha t)\cdot f(t,y_1,\cdots,y_k)\cdot dt,$$

with $\alpha=a^{1/m}$ and

$$\sigma_m(\xi)=\sum_{i=1}^m[\omega_i\exp(\omega_i\xi)]/m;$$

here $\omega_1,\cdots,\omega_m$ are the mth roots of unity;

(II) $$\partial^2 z/\partial x^2-\partial z/\partial y=f(x,y),$$

(*) $$z=\int_0^x dt\sum_{r=0}^\infty[(x-t)^{2r+1}/(2r+1)!]\partial^r f(t,y)/\partial y^r,$$

where the condition $|\partial^n f(x,y)/\partial y^n|<(Mn)^{2n}$ is sufficient for convergence. To obtain a representation of z by contour integrals the authors use the relation

$$2\pi i\sum a_r b_r=\int_C\lambda^{-1}\sum_r a_r\lambda_r^{\nu_r}\sum_r b_r\lambda_r^{-\nu_r}\cdot d\lambda.$$

The ν_r are arbitrary and chosen so as to obtain simple expressions. For equation (II) the method is applied by starting from (*) and splitting the general term of the series into two factors a_r and b_r. Because of the freedom in the choice of a_r, b_r and ν_r, a large number of contour integrals may be derived by this method, all representing the same solution of the equation. Here is an example of many of such contour integrals obtained in the paper:

$$z=\int_C(2\pi i\lambda)^{-1}d\lambda\int_0^x dt\int_t^x\exp[(x-\tau)/\lambda]f(t,y+\lambda\tau-\lambda t)d\tau,$$

which solves equation (II) with zero initial conditions.
I. Opatowski (Chicago, Ill.).

Gambier, Bertrand. Système aux dérivées partielles dont la surface de translation générale est solution. C. R. Acad. Sci. Paris 216, 244–245 (1943). [MF 10954]

Bergman, Stefan. The determination of some properties of a function satisfying a partial differential equation from its series development. Bull. Amer. Math. Soc. 50, 535–546 (1944). [MF 10846]

Let $E(z,\bar z,t)$ be an analytic function of $z=x+iy$ and $\bar z=x-iy$, and define

(1) $$u(z,\bar z)=\int_{-1}^1E(z,\bar z,t)f(z(1-t^2)/2)dt/(1-t^2)^{\frac{1}{2}}.$$

Under certain restrictions on $E(z,\bar z,t)$, $U(x,y)=\Re(u(z,\bar z))$ is a solution of

(2) $$U_{xx}+U_{yy}+A(x,y)U_x+B(x,y)U_y+C(x,y)U=0$$

for each analytic function $f(z)=\sum_{i=0}^\infty a_iz^i$. Writing $U(x,y)$ in the form

$$U(x,y)=\sum_{m,n=0}^\infty D_{mn}z^m\bar z^n,$$

the paper shows that the coefficients a_n of $f(z)$ can be expressed in terms of the sequence $\{D_{n0}\}$ and a sequence of constants depending only on the function $E(z,\bar z,t)$ and $f(0)$. Thus knowing $U(x,y)$ and $E(z,\bar z,t)$ the order of growth of $f(z)$ can be determined from classical results, since $f(z)$ is analytic. If, in addition, we know the order of growth of E, then, by use of (1), we can find the upper bounds of the order of growth of the solution $U(x,y)$ of (2). The author also

shows the role the sequence $\{D_{n0}\}$ plays in determining the type of singularity $u(z, \bar{z})$ has on its largest circle of regularity. *F. G. Dressel* (Durham, N. C.).

Tautz, Georg. Zur Theorie der elliptischen Differentialgleichungen. II. Math. Ann. 118, 733–770 (1943). [MF 10724]

In part I of this paper [Math. Ann. 117, 694–726 (1941); these Rev. 3, 126] the writer solved the Dirichlet problem for the operator

$$(1)\quad L(u) = -\int_e \frac{\partial u}{\partial n} ds + \iint_\omega \left(\frac{\partial u}{\partial x} dA + \frac{\partial u}{\partial y} dB + u dC\right)$$

for sufficiently small and regular regions. The completely additive set functions A, B, C satisfied inequalities of the form $\iint_\omega |dA| < Kr^{1+\alpha}$, $\alpha > 0$, for circles ω of radius r. In this paper it is also assumed that $C(e) \leqq 0$ and

$$\iint_e |dC| \geqq h \iint_e (|dA| + |dB|),$$

$\iint_e |dC|$, etc., denoting the variations over e.

The author begins by considering an operator $L_{f,K}(P)$ which is defined for all sufficiently small circular boundaries k in a region Ω_0, for all P inside K and all continuous f on K. For each K and for each f on K, $L_{f,K}(P)$ is continuous in P inside K and takes on the boundary values f continuously. Let Ω be a domain whose closure is in Ω_0. If $u(P)$ is continuous in Ω and $L_{u,K}(P) = u(P)$ for every K in ω, then $u(P)$ is called a "solution" in Ω. The author formulates a number of conditions which if fulfilled by this operator $L_{f,K}(P)$ will enable Perron's solution of the Dirichlet problem, among other things, to be carried through. The idea of a barrier function for a boundary point P_0 of a region Ω is introduced and its existence is shown to be necessary and sufficient for P_0 to be regular (all supposing that $L_{f,K}(P)$ satisfies the conditions above). Other criteria for the regularity of boundary points involving the notions of capacity and the "normed potential" (similar to the conductor potential) are developed. Finally it is shown that a point P_0 of the boundary of Ω is regular with respect to $L_{f,K}(P)$ if and only if it is regular for the operator Δu. The author closes by proving that the operator $L_{f,K}(P)$ defined as the solution of $L(u) = 0$ on K which takes on the boundary values f ($L(u)$ being the operator (1), restricted as in the first paragraph) satisfies all the conditions above.

C. B. Morrey, Jr. (Aberdeen, Md.).

Vekua, Ilja. On approximations to solutions of elliptic differential equations. Bull. Acad. Sci. Georgian SSR [Soobščenia Akad. Nauk Gruzinskoi SSR] 3, 97–102 (1942). (Russian. Georgian summary) [MF 10313]

By means of connections between solutions of elliptic linear differential equations and analytic functions of a complex variable, results concerning rational approximations to analytic functions [J. L. Walsh, Interpolation and Approximation by Rational Functions in the Complex Domain, Amer. Math. Soc. Colloquium Publ., v. 20, New York, 1935] are applied to obtain approximate solutions, in any given multiply-connected domain of the (x, y)-plane, of the differential equation

$$\Delta u + a(x, y)\partial u/\partial x + b(x, y)\partial u/\partial y + c(x, y)u = 0,$$

where $a(x, y)$, $b(x, y)$, $c(x, y)$ are given integral rational functions. *E. F. Beckenbach* (Austin, Tex.).

Vekua, Ilja. Solution of the fundamental boundary value problem for the equation $\Delta^{n+1}u = 0$. Bull. Acad. Sci. Georgian SSR [Soobščenia Akad. Nauk Gruzinskoi SSR] 3, 213–220 (1942). (Russian. Georgian summary) [MF 10314]

Let the finite plane domain T be bounded by the simple closed rectifiable curve L, and suppose that the Cartesian coordinates of points of L, considered as functions of the arc length s of L, are of class H_{2n}; that is, the derivatives of order $2n$ of these functions satisfy a Hölder condition. The author considers the boundary value problem of determining in T a regular solution of the equation

$$\Delta^{n+1}u = (\partial^2/\partial x^2 + \partial^2/\partial y^2)^{n+1}u = 0,$$

satisfying on L the boundary conditions

$$u = f_0(s),\ du/d\nu = f_1(s),\ \cdots,\ d^n u/d\nu^n = f_n(s),$$

where ν is normal to L and $f_k(s)$ is a given function of class H_{2n-k}, $k = 0, 1, \cdots, n$. Solutions of equation (1) are characterized as functions admitting representation of the form

$$u(x, y) = \sum_{k=0}^{n} (x^2 + y^2)^k h_k(x, y),$$

where the $h_k(x, y)$ are harmonic. A uniqueness theorem is given for the above boundary value problem and then the problem, which previously has been discussed by the author by means of conformal mapping of T on the interior of the unit circle, is solved after being reduced to an equivalent formulation in terms of Fredholm integral equations. It is stated that an analogous boundary value problem for the more general equation

$$\Delta^{n+1}u + a_1(x, y)\Delta^n u + \cdots + a_{n+1}(x, y)u = 0$$

can be solved by the same method. *E. F. Beckenbach.*

Tranter, C. J. On a problem in heat conduction. Philos. Mag. (7) 35, 102–105 (1944). [MF 10860]

The solution of the heat equation

$$\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r} + \frac{\partial^2 T}{\partial Z^2} = \frac{\partial T}{\partial t}$$

is obtained for the semi-infinite hollow cylinder $b < r < a$, $z > 0$, subject to the initial condition $T = 0$ for $t = 0$, and the boundary conditions $T = 0$ for $z = 0$, $T = 0$ for $r = a$, and $T = f(z)$ for $r = b$. The solution is obtained formally through the use of a Fourier transform followed by a Laplace transform. *F. G. Dressel* (Durham, N. C.).

de Toledo Piza, Affonso P. Solution of Fourier's partial differential equation. Anais Acad. Brasil. Ci. 16, 47–51 (1944). (Portuguese) [MF 10890]

The equation $z_{xx} + x^{-1}z_x = Kz_t$ is solved formally by

$$z = \sum_{n=0}^{\infty} K^{-n} \sum_{i=0}^{i=n} (D^2 + x^{-1}D)^i f_{n-i}(x) \cdot t^i/i!,$$

where the f's are arbitrary functions of x and $D = d/dx$.

I. Opatowski (Chicago, Ill.).

da Rocha, Miguel Mauricio. On the integration of Fourier's equation. Anais Acad. Brasil. Ci. 16, 53–56 (1944). (Portuguese) [MF 10891]

Solutions of the equation $z_{xx} + x^{-1}z_x = Kz_t$ in the form of simple polynomials of x, $\log x$, and t. *I. Opatowski.*

Minakshisundaram, S. Fourier ansatz and non-linear parabolic equations. J. Indian Math. Soc. (N.S.) 7, 129–142 (1943). [MF 10928]

The author considers the differential equation

$$\text{(A)}\qquad \frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}-\frac{\partial u}{\partial t}=f(x, y, t, u),$$

subject to the boundary conditions: (B) $u(x, y, t)=0$ on Γ $(t>0)$, and $u(x, y, 0)=u_0(x, y)$, where Γ is the boundary of a bounded closed domain G of the x, y plane. He obtains the following result. If $f(x, y, t, u)$ is a bounded function of all the variables, uniformly continuous with respect to u, and such that $|f(x, y, t, u)|\leqq\varphi(t)$, $\varphi(0)=0$, for some positive nondecreasing function $\varphi(t)$, then there exists a solution of (A) satisfying (B). The equation

$$\text{(A')}\qquad \frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}-\frac{\partial u}{\partial t}=f(x, y, t, u, p, q),$$

where $p=\partial u/\partial x$ and $q=\partial u/\partial y$, is also discussed. If $f(x, y, t, u, p, q)$ is a continuous function of all the variables, vanishing on Γ, and if $|f(x, y, t, u, p, q)|\leqq\varphi(t)$, $\varphi(0)=0$, where $\varphi(t)$ is some positive nondecreasing function of t, then there exists a solution of (A′) satisfying (B). The author notes that the results may be extended to the corresponding problems for similar equations in n independent variables.

F. W. Perkins (Hanover, N. H.).

Abdelhay, J. On a special partial differential equation. Anais Acad. Brasil. Ci. 16, 139–141 (1944). (Portuguese) [MF 11005]

Laplace's transformation is used to find the well-known solution of the differential equation $u_{xx}-u_{yy}=0$ subject to the initial conditions $u(x, 0)=f(x)$, $u_y(x, 0)=h(x)$.

F. G. Dressel (Durham, N. C.).

Cinquini-Cibrario, Maria. Sul problema di Goursat per le equazioni del tipo iperbolico non lineari. Ann. Mat. Pura Appl. (4) 21, 189–229 (1942). [MF 10510]

The author considers the hyperbolic equation

$$\text{(1)}\qquad F(x, y, z, p, q, r, s, t)=0$$

$(F_s^2-4F_rF_t>0)$ for the real function $z=z(x, y)$. Here p, q and r, s, t denote, respectively, the first and second partial derivatives of z. The "problem of Goursat" consists in finding a solution z taking given values on two arcs of curves γ_1 and γ_2 in the xy-plane. A solution had been given previously by E. E. Levi [Rend. Accad. Naz. Lincei (5) 17, 331–339 (1908)]. The solution given in the present paper uses the methods applied by H. Lewy in solving the Cauchy problem [Math. Ann. 98, 179–191 (1928); see also the paper by Friedrichs and Lewy, Math. Ann. 99, 200–221 (1929)].

The author succeeds in reducing (1) to a system Σ of first order quasi-linear differential equations, to which the method of successive approximations can be applied. This reduction is achieved by simplifying the boundary conditions, and by using the characteristic curves on the prospective integral surface as coordinate curves for a system of curvilinear coordinates λ, μ. Then Σ is a system of equations for x, y, z, p, q, r, t as functions of λ, μ.

The following results are proved. Let F be of class C^9 and the arcs γ_1 and γ_2 be given by equations of class C^5. These arcs shall intersect in a point O interior to both. Let values of z of class C^5 be prescribed on both arcs in a neighborhood of O. Let γ_1 and γ_2 have distinct tangents at O, which are not divided harmonically by the projections on the xy-plane of the characteristic directions at O of the integral surface (those directions are determined by (1) and by the given values of z on γ_1 and γ_2). Then there is a neighborhood of O in which there exists and is uniquely determined a solution of (1) taking the given values. In the case where γ_1 and γ_2 are arcs having O as common endpoint, existence and uniqueness of z for given values on γ_1 and γ_2 can be guaranteed under analogous conditions in points sufficiently close to O in the angle formed by γ_1 and γ_2, provided that angle is free of projections of the characteristic directions.

F. John (Aberdeen, Md.).

Theory of Probability

Münzner, H. und Schwarz, H. Ein Zusammenhang zwischen Erneuerungszahlen und dem Moivre'schen Problem. Arch. Math. Wirtsch.-Sozialforschg. 6, 46–49 (1940). [MF 10652]

The authors point out that the problem of self-renewing aggregates can be formulated by means of de Moivre's classical urn scheme. For that purpose it is only necessary to assume that the time each element belongs to the aggregate is a multiple of a certain unit time. The paper contains no results.

W. Feller (Providence, R. I.).

Raimondi, Elba R. On a problem of geometric probabilities on the sets of triangles. Union Mat. Argentina, Publ. no. 25, 18 pp. (1942). (Spanish) [MF 11022]

Identical with a paper in Revista Union Mat. Argentina 8, 1–16 (1942); these Rev. 3, 302.

Landau, H. G. Note on the variance and best estimates. Ann. Math. Statistics 15, 219–221 (1944). [MF 10831]

The author proves the following theorem: if the random variables x_1 and x_2 have finite variances σ_1^2 and σ_2^2, and $\sigma_1^2\leqq\sigma_2^2$, then either $Q(t)=P_1(t)-P_2(t)$ is equal to zero at all points of continuity, which can occur only for $\sigma_1^2=\sigma_2^2$, or there is an interval $t_1<t<t_2$ in which $Q(t)$ is positive, where

$$P_i(t)=\text{Prob}\,\{|x_i-E(x_i)|<t\},\qquad i=1, 2.$$

D. Blackwell (Washington, D. C.).

Hornich, Hans. Zur Theorie des Risikos. Monatsh. Math. Phys. 50, 142–150 (1941). [MF 10478]

The paper is written in actuarial terminology. Translated into the language of calculus of probabilities, the results read as follows. Let x be a random variable with symmetrical distribution, and let $x_1, x_2, \cdots, x_s$ be independent repetitions of x. If $D^{(1)}$ denotes the expectation of $|x|$ and $D^{(s)}$ the expectation of $|x_1+x_2+\cdots+x_s|$ then, for s odd, one obtains the inequality

$$D^{(s)}\geqq\frac{s}{2^{s-1}}\binom{s-1}{(s-1)/2}\cdot D^{(1)}$$

and, for s even, the inequality

$$D^{(s)}\geqq\frac{s}{2^{s-1}}\binom{s-1}{s/2-1}\cdot D^{(1)}.$$

If the assumption of symmetry is replaced by the more general condition $E(x)=0$, then the lower bounds in those equations are divided by 2. An application of Stirling's formula shows that $D^{(s)}$ increases like $\sqrt{s}$.

Z. W. Birnbaum (Seattle, Wash.).

Wolfowitz, J. Note on runs of consecutive elements. Ann. Math. Statistics 15, 97–98 (1944). [MF 10243]

Let $R=(x_1, \cdots, x_n)$ be a random permutation of the first n positive integers, where each permutation has the same probability $1/n!$. Denote by $W(R)$ the total number of runs in R. It is shown in this note that $n-W(R)$ has in the limit, as n approaches infinity, the Poisson distribution with mean value 2. *A. Wald* (New York, N. Y.).

Halmos, Paul R. Random alms. Ann. Math. Statistics 15, 182–189 (1944). [MF 10825]

Let $x_0, x_1, x_2, \cdots$ be random variables such that x_0 $(0\leqq x_0\leqq 1)$ has probability density $p(\lambda)$ and the following formula for conditional probabilities holds:

$$P\{a<x_n<b\,|\,\sum_{j<n} x_j=1-\mu\}=\int_a^b (1/\mu)p(\lambda/\mu)d\lambda,\quad 0\leqq a<b\leqq\mu.$$

(The author is led to this problem by considering the question of distributing a pound of gold dust among infinitely many beggars who consecutively take random amounts of gold dust.) The author then studies in detail the distributions of x_n and $S_n=\sum_{j<n}x_j$ and also the "average" rate of convergence of the series $\sum x_j$. All these questions are encountered in calculating the energy loss of a neutron due to collisions with protons. *M. Kac* (Ithaca, N. Y.).

Nolfi, P. Wahrscheinlichkeit unstetiger Vorgänge bei kontinuierlich wirkenden Ursachen. Comment. Math. Helv. 15, 36–44 (1943).

The author considers random events which (like deaths) may be uniquely attributed to one of the categories $E_1, \cdots, E_k$. He uses a conventional urn scheme, but his assumptions are trivially equivalent to assuming that (1) the total number of events in time t is distributed according to the Poisson law and (2) the polynomial law with probabilities $p_1, \cdots, p_k$ regulates the distribution of the events among the E_i. The author finds this result after much unnecessary computation and hails it as an important "generalization" of the Poisson law. He fails to notice that his passage to the limit rests on arbitrary assumptions. It seems more natural to assume that the number of events in each E_i obeys a Poisson law, in which case the total number of events also obeys the same law. *W. Feller* (Providence, R. I.).

Ruark, Arthur E. Differential equations for the probability distribution of events. Phys. Rev. (2) 65, 88–90 (1944). [MF 10086]

Generalizing the classical case of random events distributed according to the Poisson formula, the author considers a problem "the general features of which can be well understood from a simple example. Suppose we have a piece of cloth, of length x and width y; let $f(x, y)dxdy$ be the chance that a flaw occurs in the element $dxdy$. Then the chance that there are n flaws in the area xy may be called $W_n(x, y)$ and we desire a set of differential equations from which we can find W_0, W_1, etc." Putting $X=\int_0^y f dy$ and $Y=\int_0^x f dx$, a straightforward argument shows that $\partial W_n/\partial x=XW_{n-1}-XW_n$ and $\partial W_n/\partial y=YW_{n-1}-YW_n$. The more general case where the coefficients depend on n is also considered and the integration discussed. *W. Feller* (Providence, R. I.).

Lévy, Paul. Un théorème d'invariance projective relatif au mouvement brownien. Comment. Math. Helv. 16, 242–248 (1944).

Let $\{x(t)\}$ be the usual chance variables associated with the Brownian movement: $x(t)-x(s)$ is a Gaussian variable with mean 0 and dispersion $|t-s|$. In a previous paper [Compositio Math. 7, 283–339 (1939); these Rev. 1, 150] the author discussed various properties (such as the vanishing) of $x(t)$ in an interval (b, c) under the hypothesis $x(a)=0$ $(a<b<c)$. It is shown in the present paper how to deduce from these theorems corresponding theorems under the hypothesis $x(a)=x(d)=0$ $(a<b<c<d)$. Thus, if $x(a)=x(d)=0$, the probability that $x(t)$ does not vanish in the interval $b\leqq t\leqq c$ is

$$(2/\pi)\arcsin[(b-a)(d-c)/(c-a)(d-b)]^{\frac{1}{2}}.$$

J. L. Doob (Washington, D. C.).

Cameron, R. H. and Martin, W. T. Transformations of Wiener integrals under translations. Ann. of Math. (2) 45, 386–396 (1944). [MF 10276]

The authors discuss certain transformations on the space of continuous functions $x(t)$. The measure used on this space was first defined rigorously by Wiener; in probability language, (1) $x(t+h)-x(t)$ is a chance variable with a Gaussian distribution, having mean 0 and dispersion $|h|$; (2) $x(0)=0$; (3) if $t_1<\cdots<t_n$, $x(t_2)-x(t_1), \cdots, x(t_n)-x(t_{n-1})$ are mutually independent. A simple formula is derived for the measure of the image of a set under the transformation $y(t)=x(t)+x_0(t)$, where $x_0(t)$ is a fixed continuous function vanishing at the origin. The formula and simple consequences can be used in evaluating certain integrals considered by Wiener [cf. Paley and Wiener, Fourier Transforms in the Complex Domain, Amer. Math. Soc. Colloquium Publ., v. 19, New York, 1934, chap. 9, 10]. [In lemma 3 the condition should be stated as a condition for integrability rather than for measurability.] *J. L. Doob.*

Robbins, H. E. On the measure of a random set. Ann. Math. Statistics 15, 70–74 (1944). [MF 10239]

Let T be a space of Lebesgue measurable sets X (X is in the n-dimensional Euclidean space E_n) and suppose that a probability measure over T is defined. Suppose furthermore that the function $g(x, X)$, which is 1 if $x\epsilon X$ and 0 otherwise, is a measurable function of the pair (x, X). Then the average (first moment) of the measure of X is equal to the Lebesgue integral over E_n of $p(x)=\text{Prob}\,(x\epsilon X)$. Similar theorems for higher moments are obtained and the general result is illustrated by examples. *M. Kac.*

van der Velden, H. A. and Endt, P. M. On some fluctuation problems connected with the counting of impulses produced by a Geiger-Müller counter or ionisation chamber. Physica 9, 641–657 (1942). [MF 9823]

The fluctuation problems considered concern an electric counting rate meter, which is an electrical device used for the counting of the number of kicks which come from a Geiger counter. In particular, the sensibility of the meter is investigated and compared with that of mechanical recorders. The methods are similar to those used in the theory of Brownian motion. *W. Feller.*

Mathematical Statistics

Lorenz, Paul. Darstellung statistischer Übersichten mit zwei Eingängen durch orthogonale ganze rationale Funktionen (Flächendarstellung). Arch. Math. Wirtsch.-Sozialforschg. 6, 57–70 (1940). [MF 10648]

Sawkins, D. T. Simple regression and correlation. J. Proc. Roy. Soc. New South Wales 77, 85–95 (1944). [MF 10175]

Albert, A. A. The minimum rank of a correlation matrix. Proc. Nat. Acad. Sci. U. S. A. **30**, 144–146 (1944). [MF 10743]

The ideal rank r of a correlation matrix is defined as the largest order of a nonvanishing minor which does not contain any diagonal elements. Let D be a diagonal matrix such that $M+D$ has minimum rank. The diagonal elements of D are called communalities. By a proper choice of the notation it is always possible to write M in the form

$$R=\begin{pmatrix} A & B' & G' \\ B & C & H' \\ G & H & K \end{pmatrix},$$

where A, B, C are r-rowed square matrices and B is nonsingular. The author proves the following theorem. If the ranks of G and H are equal to the ideal rank r of R then the communalities are uniquely determined and the minimum rank of $R+D$ is r. The author also shows that in order that the minimum rank of $R+D$ be equal to the ideal rank of R it is necessary but not sufficient that the rank of G equal the rank of H. The author's result can be applied to any symmetric matrix. *H. B. Mann* (Columbus, Ohio).

Guttman, Louis. General theory and methods for matric factoring. Psychometrika **9**, 1–16 (1944). [MF 10233]

By a generalization of the Lagrange method for the reduction of bilinear and quadratic forms, it is shown how an arbitrary rectangular matrix S of rank r can be factored into the form FP in which F has r columns and P r rows and in which F is built up s columns and P s rows at a step. If S is the matrix of scores on a battery of tests, then F is a matrix of "factor" loadings, P is a matrix of "factor" scores and r is the number of "factors." The same method is also adopted for factoring an arbitrary Gramian G of rank r, which in applications may be the matrix of intercorrelations of a battery of tests, into the form FF' in which F has r columns. Moreover, SS' is proportional to R and it is shown that S may be factored by factoring R, but it is also possible to factor S directly in as many stages as is desired though it may not be convenient to remove more than three "factors" in each step. Thurstone's centroid method is justified as a special case of this procedure and it is pointed out that one does not need to be restricted to vectors of the type $\|\pm1, \pm1, \cdots, \pm1\|$ in the centroid method and that a judicious choice of other elements can more rapidly reduce the residual variance. *C. C. Craig.*

Thurstone, L. L. Graphical method of factoring the correlation matrix. Proc. Nat. Acad. Sci. U. S. A. **30**, 129–134 (1944). [MF 10741]

The author presents a graphical method of reducing an $n\times n$ correlation matrix R to an $n\times r$ factor matrix F, where r is the rank of R and $R=FF'$. This iterative method is claimed to be shorter for a rather large n than that presented by Hotelling [J. Educ. Psych. **24**, 417–441, 498–520 (1933)]. An example is given for $n=10$, $r=3$ and correlations carried to two decimal places. *R. L. Anderson.*

Dixon, Wilfrid J. Further contributions to the problem of serial correlation. Ann. Math. Statistics **15**, 119–144 (1944). [MF 10821]

This article furnishes new contributions to the solution of the expanding problems of serial correlation and introduces some useful tools for future research. The author considers distributions related to the following quantities:

$$\delta_n^2=\sum_1^n (x_i-x_{i+1})^2, \quad C_n=\sum_1^n (x_i-\bar{x})(x_{i+1}-\bar{x}),$$

$$_lC_n=\sum_1^n (x_i-\bar{x})(x_{i+l}-\bar{x}), \quad v_n=\sum_1^n (x_i-\bar{x})^2,$$

where $x_{n+i}=x_i$, the x's are normally and independently distributed with mean a and variance σ^2 and l is the lag considered.

Using the relationship $x_\alpha=a+bx_{\alpha-l}$, the likelihood criterion to test $H_1:b=0$ is $\lambda_1=(1-\hat{b}^2)^{\frac{1}{2}n}$, where $\hat{b}={}_lC_n/v_n$, and that to test $_0H_1(a=0):b=0$ is $_0\lambda_1=(1-\hat{b}_0^2)^{\frac{1}{2}n}$, where $\hat{b}_0=\sum x_\alpha x_{\alpha+l}/\epsilon x_\alpha^2$. Approximations to the moment generating function are used to derive the moments of $\hat{b}$ and $\hat{b}_0$; these are exact for moments less than $2n/\alpha$, where α is the greatest common factor of l and n. These moments are used to derive the distribution functions of $\hat{b}$, $\hat{b}_0$, $\lambda_1^{2/n}$ and $_0\lambda_1^{2/n}$, which are quite simple. For example, the distribution of $\hat{b}_0=x$ is $K_1(1-x^2)^{\frac{1}{2}(n-1)}$. For the distribution of $\hat{b}$, a Pearson type I approximation is used to determine 1% and 5% positive and negative significance levels. All of these are exact, to the three decimal places used, for $n\geqq25$ and all but the positive 1% value within 0.001 at $n=15$ (positive 1% within 0.002). The 5% values are even within 0.002 of the exact values at $n=10$. The moments and distribution are also obtained for $_0\eta_1=\delta_n^2/\sum x_i^2=2(1-\hat{b}_0)$ and for $\eta_1=\delta_n^2/v_n$; also the first 2 moments for

$$\eta_2=\sum_1^n (x_i-2x_{i+1}+x_{i+2})^2/v_n.$$

The general λ-criterion is set up to test $H_{r,m}:b_{m+1}, \cdots, b_r=0$ for $x_\alpha=a+\sum_{i=1}^r b_i x_{\alpha-l_i}$. The mean and variance are given for $r=2$; $m=0, 1$ ($a=0$ and $a\neq0$). Finally, the set-up for the serial correlation in several variables is indicated but not solved. *R. L. Anderson* (Princeton, N. J.).

Gebelein, Hans. Verfahren zur Beurteilung einer sehr geringen Korrelation zwischen zwei statistischen Merkmalsreihen. Z. Angew. Math. Mech. **22**, 286–298 (1942). [MF 8914]

This paper discusses the problem of testing for independence in contingency tables. The author uses first the combinatorial approach: given row and column totals, the distribution of cell frequencies is determined and the first two moments found. Then the asymptotic theory is taken up in the usual manner. A similar discussion in English may be found in Wilks' "Mathematical Statistics" [Princeton University Press, Princeton, N. J., 1943; these Rev. **5**, 41], where the combinatorial method is employed.

The author also considers (1) the testing of the difference between two cell frequencies in the same row or column of a contingency table and (2) the testing of the significance of the sum of the main diagonal elements of a square contingency table. For these tests the author computes the mean and variance and would presumably use the normal distribution to get significance levels. *A. M. Mood.*

Hoel, Paul G. On statistical coefficients of likeness. Univ. California Publ. Math. (N.S.) **2** [No. 1, Seminar Rep. in Math. (Los Angeles)], 1–8 (1944). [MF 10451]

Let $x_{i\alpha k}$ be the measurement of character i for individual α in sample k ($i=1, \cdots, n$; $k=1, \cdots, p$; $\alpha=1, \cdots, m$), where the n character measurements follow the usual normal multinomial distribution. After assuming no cor-

relation between the x_i's, Karl Pearson [Biometrika **18**, 105–117 (1926)] defined a coefficient of racial likeness between two samples with means $\bar{x}_i$ and $\bar{x}_i'$ to be z/n, where

$$z=\frac{m}{2}\sum_{i=1}^{n}\left(\frac{\bar{x}_i-\bar{x}_i'}{\sigma_i}\right)^2.$$

The present paper first considers the distribution of z when the x_i's are correlated. The hth seminvariant is

$$2^{h-1}(h-1)!\sum_{q=1}^{n}\lambda_q^h,$$

where the λ's are the characteristic roots of the correlation matrix (ρ_{ij}). An example is given with eight character measurements whose mean correlation was 0.44, showing the unsatisfactory nature of the Pearson assumption of zero correlations.

Using a result of Wilks [Biometrika **24**, 471–494 (1932)], the distribution of

$$z'=(m/2)\sum_{i,j=1}^{n}A'_{ij}(\bar{x}_i-\bar{x}_i')(\bar{x}_j-\bar{x}_j')$$

is derived, where A'_{ij} is the sample estimate of A_{ij} and $A^{ij}=\rho_{ij}\sigma_i\sigma_j$. In fact, $2m/(z'+2m)$ is distributed as W, which, for two samples ($p=2$), has the distribution

$$f(W)=CW^{\{(2m-n-1)/2\}-1}(1-W)^{(n/2)-1},$$

where W tests the hypothesis of equal sample means, assuming equal covariances. It is also shown that this test is the same as Fisher's "z" test, where

$$\text{"}z\text{"}=\frac{1}{2}\left\{\log\frac{2m-n-1}{n}+\log\left(\frac{1}{W}-1\right)\right\}.$$

R. L. Anderson (Princeton, N. J.).

Cole, R. H. Associated frequency distributions in biometry. Amer. Math. Monthly **51**, 252–261 (1944). [MF 10563]

A set of spheres imbedded in a three dimensional space with their radii falling in the range (a, b) is cut at random by planes, producing a sample of circular areas. These areas are subdivided into n groups at the equally spaced points $x_j^2=jw$ ($j=0, 1, \cdots, n$), where $w=b^2/n$. The radii, represented by x, have the probability function $\Phi(x)$; the relative frequency of x on the interval (x_{j-1}, x_j) is F_j, where $\Phi(x)=F_j/(x_j-x_{j-1})$. The probability function of the sample is $cx\Phi(x)$.

If the relative frequency of an area y on the interval (y_{i-1}, y_i) is f_i, then

$$f_i=\sum_{j=1}^{n}a_{i,j}F_j, \qquad i=1, 2, \cdots, n, \tag{1}$$

where $a_{i,j}=c(I_{i-1,j}-I_{i,j})$ and

$$\begin{aligned} I_{i,j}&=\frac{1}{x_j-x_{j-1}}\int_{x_{j-1}}^{x_j}(x^2-x_i^2)^{\frac{1}{2}}dx, && i<j,\\ I_{i,j}&=0, && i\geqq j. \end{aligned}\tag{2}$$

Then $I_{i,j}$ is approximated by use of the trapezoidal rule so that

$$I_{i,j}\doteqdot(\omega^{\frac{1}{2}}/2m^{\frac{1}{2}})\Big[\{m(j-i-1)\}^{\frac{1}{2}} +2\sum_{r=1}^{m-1}\{m(j-i-1)+r\}^{\frac{1}{2}}+\{m(j-i)\}^{\frac{1}{2}}\Big]; \tag{3}$$

c is determined so that $a_{n-k,n-k}=1$. We obtain:

$$F_{n-k}=f_{n-k,k}, \qquad k=0, 1, \cdots, n-1;$$

$$f_{i,k+1}=f_{i,k}-R_{i,k}f_{n-k,k}$$

for $k=0, 1, \cdots, n-2$ and $i=1, 2, \cdots, n-k-1$ ($f_{i,0}=f_i$);

$$R_{i,k}=a_{i,n-k}. \tag{4}$$

Comparing (2), (3) and (4), we see that $R_{n-k-i,k}$ is constant for all k. Values of $R_{n-k-i,k}$ have been computed for $i=1, 2, \cdots, 15$, using $m=3$, and the sphere frequencies computed for a problem with 5 area classes. It is later shown that the same methods could be extended to approximate the frequencies for nonspherical structures. [It might be mentioned that the author could have derived the limiting value of $I_{n-k-i,n-k}$ from (3) to be as follows:

$$\lim_{m\to\infty} I_{n-k-i,n-k}\doteqdot 2\omega^{\frac{1}{2}}[i^{\frac{3}{2}}-(i-1)^{\frac{3}{2}}]/3.$$

Hence $c=3/2\omega^{\frac{1}{2}}$, $cI_{n-k-i,n-k}=[i^{\frac{3}{2}}-(i-1)^{\frac{3}{2}}]$ and $R_{n-k-i,k}=(i+1)^{\frac{3}{2}}-2i^{\frac{3}{2}}+(i-1)^{\frac{3}{2}}$.]

R. L. Anderson (Princeton, N. J.).

Bhattacharyya, A. On a measure of divergence between two statistical populations defined by their probability distributions. Bull. Calcutta Math. Soc. **35**, 99–109 (1943). [MF 10737]

Given two multinomial populations P and P', with probabilities $\{\pi_i\}$ and $\{\pi_i'\}$, where $\sum_1^k\pi_i=\sum_1^k\pi_i'=1$, represented in k-space by population lines with direction cosines $\sqrt{\pi_i}$ and $\sqrt{\pi_i'}$. The angle of divergence Δ is such that $\cos\Delta=\sum_1^k(\pi_i\pi_i')^{\frac{1}{2}}$. Take a sample of n from P with frequencies $\{n_i\}$ and define $p_i=n_i/n$, where $\lim_{n\to\infty}p_i=\pi_i$. This gives a sample line with direction cosines $\{\sqrt{p_i}\}$. The divergence angle D between the sample and the population lines, for n large, is approximately $\chi/2\sqrt{n}$.

If P and P' are binomial or Poisson distributions, $-\log\cos\Delta'$ can be used as a measure of divergence for n large, where $-\log\cos\Delta'$ equals $-n\log\cos\Delta$ for binomial and $(\sqrt{m}-\sqrt{m'})^2/2$ for Poisson distributions. Using the normal approximation to the binomial with variance $\pi(1-\pi)/n$ and to the Poisson with $\sqrt{x}$ distributed normally about $\sqrt{m}$, we obtain the same results as above.

For two normal populations with $\sigma_1=\sigma_2=\sigma$, the divergence measure is $-8\log\cos\Delta=(m_1-m_2)^2/\sigma^2$, estimated by

$$\frac{(\bar{x}_1-\bar{x}_2)^2}{(n_1s_1^2+n_2s_2^2)/(n_1+n_2-2)}=t^2.$$

If $m_1=m_2$, the measure of divergence is σ_1^2/σ_2^2 or $\log(\sigma_1/\sigma_2)$, estimated by s_1^2/s_2^2 or $\log(s_1/s_2)$. Finally, if σ_1/σ_2 is a constant, the measure of divergence is $(m_1-m_2)^2/(\sigma_1^2+\sigma_2^2)$, estimated by $(\bar{x}_1-\bar{x}_2)^2/(s_1^2+s_2^2)$. If $\log s$ is normally distributed about $\log\sigma$ with variance $\frac{1}{2}n$, one obtains the same measure of divergence for the standard deviations of samples of n_1 and n_2.

Similar results are also given for two multivariate normals and n multinomial populations, the latter giving the same measure of divergence as the analysis of variance for univariate normals with only the means being unequal. Also, the measure of divergence for a continuous population can be represented as an arc angle between two points on the surface of a hypersphere and this connected with Fisher's intrinsic accuracy. *R. L. Anderson* (Princeton, N. J.).

Bhattacharyya, B. C. On an aspect of Pearsonian system of curves and a few analogies. Sankhyā 6, 415–418 (1944). [MF 10623]

Some examples of the Pearsonian curves are developed in terms of χ^2, such as the type I distribution in terms of $z=\chi_1^2/(\chi_1^2+\chi_2^2)$. The function $f(z)=ke^{-az}\cos^{2m-1}z$ is developed in terms of $e^{-x/2}$, where x is distributed as χ^2 with 2 d.f. For $m=1$, $x=A+2az$; for $m=2$, $x=B+2az-2\log[-a\cos z+\sin z]$; etc. The distributions of certain useful combinations of χ_1^2(f_1 d.f.) and χ^2(f_2 d.f.) are given.

R. L. Anderson (Princeton, N. J.).

Bhattacharyya, A. On some sets of sufficient conditions leading to the normal bivariate distribution. Sankhyā 6, 399–406 (1944). [MF 10620]

Let $f(x, y)$ be a probability distribution in variables x, y. The following conditions are well known if f is normal. (1_x) The array distribution in x, $f_1(x)=f(x, y)/\int f(x, y)dy$, is normal. (By ($1_y$) we shall mean the analogous condition for y.) (2_x) The variance of the array distributions in x is constant. (3_x) The regression curve $x=\int xf_1(x)dx$ is linear. (4_x) The marginal distribution in x, that is, $\int f(x, y)dy$, is normal. (5) The equiprobable contours are similar concentric ellipses. (6) Every linear function of x and y is normally distributed. (7) For any linear function of x and y there is a second linear function such that the two are statistically independent. The author displays nine sets of sufficient conditions: (I) 1_x, 5; (II) 1_x, 2_x, 3_x, 4_y; (III) 1_x, 1_y, 2_x; (IV) 1_x, 1_y, 4_x; (V) 1_x, 1_y, 3_x (with slope not equal to 0); (VI) 4_x, 5; (VII) 2_x, 5; (VIII) 6; (IX) 7. The methods are for the most part elementary, but Fourier inversion is used to evaluate certain characteristic functions.

I. Kaplansky (New York, N. Y.).

Sukhatme, P. V. Moments and product moments of moment-statistics for samples of the finite and infinite populations. Sankhyā 6, 363–382 (1944). [MF 10618]

The author uses results from his earlier paper on bipartitional functions [Philos. Trans. Roy. Soc. London. Ser. A. **237**, 375–409 (1938)] to calculate and record moments and product moments of central moments in samples from finite populations of weight 8 or less with the exception of those of weight 8 in which the partition of 8 is of 2 parts or more and one part is greater than 4. The procedure is to expand a sample moment into products of power sums and use the tables of the earlier paper to find first the expected values in terms of monomial symmetric functions in the universe which are then expressed in terms of central moments in the universe. The results are left with coefficients which are linear forms in the polynomials $e_i=n^{(i)}/N^{(i)}$, n being the sample size and N the size of the universe. The reader will note the close resemblance of this method to that of Dwyer [Ann. Math. Statistics **8**, 21–65 (1937); **9**, 1–47, 97–132 (1938)] which was apparently unknown to Sukhatme.

C. C. Craig (Ann Arbor, Mich.).

Wolfowitz, J. Asymptotic distribution of runs up and down. Ann. Math. Statistics **15**, 163–172 (1944). [MF 10823]

This paper contains three results, the first of which (that runs up and down are asymptotically normally distributed) has been awaited for several years because it is not subject to ordinary direct methods of attack. Specifically, they are the following. (1) Let $h_1, \cdots, h_n$ be a permutation of n unequal numbers, and let the probability of any given permutation be $1/n!$. This will be the case when the n numbers are drawn at random without replacement, or when a sample of size n is drawn from any continuous population. A sequence of p successive $+(-)$ signs not immediately preceded or followed by a $+(-)$ sign in the sequence of signs of the first differences $h_{i+1}-h_i$ is called a run up (down) of length p. Let s_p be the number of runs up of length p, and s_p' be the number of runs up of length p or more; t_p and t_p' are similarly defined for runs down. The author proves that the variables $s_1, \cdots, s_k, s'_{k+1}, t_1, \cdots, t_m$ are asymptotically jointly normally distributed for any k and m. The covariance matrix has not yet been obtained. However, Levene and Wolfowitz [Ann. Math. Statistics **15**, 58–69 (1944); these Rev. **5**, 208] have provided the covariance matrix for the variables $r_1=s_1+t_1, \cdots, r_k=s_k+t_k$, $r'_{k+1}=s'_{k+1}+t'_{k+1}$, which are of course asymptotically normally distributed. (2) Let $f(i)$ be defined for $i=1, 2, 3, \cdots$ such that: (a) there exists a pair of integers a and b for which $bf(a)\neq af(b)$; (b) for any $\epsilon>0$ there exists a positive integer N_ϵ such that, for all $n\geqq N_\epsilon$,

$$\sum_{i=N_\epsilon}^{n-1}|f(i)|\sigma(r_i)<\epsilon n.$$

Then $F=\sum_{i=1}^{n-1}f(i)r_i$ is asymptotically normally distributed. (3) In quality control, data are often accumulated over a period of time; at any given time a significantly long run must have its length defined in terms of the amount of data collected at that time. With reference to this problem the author proves the following theorem. Let p vary so that $n/(p+1)!=K$, a fixed positive number. Then

$$\lim_{n\to\infty}\text{Prob}\,(r_p=j)=e^{-2K}(2K)^j/j!,\quad j=0, 1, 2, \cdots,$$

that is, r_p is asymptotically distributed by the Poisson distribution with mean $2K$.

A. M. Mood.

Ville, Jean. Sur l'application, à un critère d'indépendance, du dénombrement des inversions présentées par une permutation. C. R. Acad. Sci. Paris **217**, 41–42 (1943). [MF 10633]

Assuming that all arrangements are equally likely, the author finds the probability that a permutation of $(1, 2, \cdots, n)$ will exhibit k inversions. He suggests a test of statistical independence quite similar to the tests based on runs which are familiar to quality control engineers. These simpler tests seem unknown to the author.

W. Feller.

Tippett, L. H. C. The control of industrial processes subject to trends in quality. Biometrika **33**, 163–172 (1944). [MF 10897]

This paper treats a problem in the control of quality of manufactured articles which can be formulated as follows. Let $\bar{X}=d+ah$ be the expected value of a normally distributed chance variable which denotes the quality of a product produced at time h. Here $\bar{X}$ corresponds to the characteristic of a tool which produces the article and which deteriorates linearly (d and a are constants); large values (say) denote poor quality. The inspection procedure is such that, when the observed value of a unit of the product or the mean of a number of consecutive units pooled into one sample exceeds d, the tool is discarded. It is assumed that the interval between the time zero (when the tool has the value d) and the time of the next observed sample is uniformly distributed with range equal to the interval between successive samples. The standard deviation of the observed characteristic of quality is assumed to be a constant multiple k of a, the rate of deterioration of the tool.

On the basis of this scheme the author derives a recursion formula with the aid of which he computes several interesting tables. Table 1 gives, for a number of values of k, the probability that a tool will be discarded after various times. Table 2 gives the mean time of discard. Several related functions are graphed to facilitate the evaluation of d in any particular problem, of which two are solved as illustrations. The question of optimum sample size is briefly discussed and the relation of this work to Shewhart control ~~cards~~ charts touched upon. *J. Wolfowitz.*

Robbins, Herbert. On distribution-free tolerance limits in random sampling. Ann. Math. Statistics 15, 214–216 (1944). [MF 10828]

The author has proved, essentially, a conjecture of Wilks, that the only distribution free statistics are order statistics. He shows the following. (1) If $x_1, \cdots, x_n$ are random independent variables with the same continuous and differentiable cumulative distribution function $\sigma(x)$, then a necessary and sufficient condition that the continuous function $f(x_1, \cdots, x_n)$ be distributed independently of $\sigma(x)$ is that the function

$$f(x_1, \cdots, x_n) = \prod_{i=1}^{n} \{f(x_1, \cdots, x_n) - x_i\}$$

be identically zero. (2) If $f(x_1, \cdots, x_n)$ satisfies $\bar{f}=0$ and is symmetric (that is, unchanged by permutations of its arguments), then it must be one of the n functions $O_r(x_1, \cdots, x_n)$, $r=1, \cdots, n$, where $O_r(x_1, \cdots, x_n)$ is a function whose value is the rth term when the numbers $x_1, \cdots, x_n$ are arranged in nondecreasing order of magnitude. *A. M. Mood* (Princeton, N. J.).

Scheffé, H. and Tukey, J. W. A formula for sample sizes for population tolerance limits. Ann. Math. Statistics 15, 217 (1944). [MF 10829]

The authors present an approximate formula for the calculation of sample sizes for Wilks' population tolerance limits. Let $x_1, \cdots, x_n$ be a sample of size n from a population with cumulative distribution function $F(x)$ arranged in nondecreasing order of magnitude. The sample size n for which

$$\text{Prob}\left[\int_{x_k}^{x_{n-m+1}} dF \geqq b\right] = 1-\alpha$$

is determined by $I_b(n-r+1, r)=\alpha$, where the term on the left is the incomplete beta function in Karl Pearson's notation and $r=k+m$. The approximate solution is

$$n \doteq \tfrac{1}{4}\chi_\alpha^2(1+b)/(1-b)+\tfrac{1}{2}(r-1),$$

where χ_α^2 is the 100α percent point on the χ^2-distribution with $2r$ degrees of freedom. The derivation of this approximate formula is promised in a later paper. For $\alpha \leqq 0.1$ and $b \geqq 0.9$, "extensive numerical calculations indicate that the error is less than 0.1%." *A. M. Mood.*

Scheffé, Henry. Note on the use of the tables of percentage points of the incomplete beta function to calculate small sample confidence intervals for a binomial p. Biometrika 33, 181 (1944). [MF 10899]

Bliss, C. I. A chart of the chi-square distribution. J. Amer. Statist. Assoc. 39, 246–248 (1944). [MF 11204]

Gumbel, E. J. On the reliability of the classical chi-square test. Ann. Math. Statistics 14, 253–263 (1943). [MF 9143]

The author notes that the application of the classical χ^2-test for the goodness of fit of an observed to a theoretical (continuous) distribution requires three arbitrary choices: (1) number and size of class intervals, (2) starting point of first interval, (3) combination of intervals with small expected frequencies. He devotes most of the paper to a numerical example which should interest and disturb statisticians. Using previously published data, and varying only choice (2) in a natural way, he finds that four different starting points give the "probability of χ^2" the values 0.023, 0.057, 0.399, 0.705. In conclusion he recommends the use of the transformation taking the theoretical distribution into the uniform distribution before application of the test. *H. Scheffé* (Syracuse, N. Y.).

Simpson, Harold. On a theorem concerning sampling. J. Roy. Statist. Soc. (N.S.) 106, 266–267 (1943). [MF 10735]

Consider i samples from a normal population containing $n_1, \cdots, n_i$ variates, respectively. Let m_k be the mean of the kth sample and s_k^2 its variance. Finally, let $a_1, \cdots, a_i$ be constants such that $\sum(a_k^2/n_k)=1$. The author shows that the probability that

$$a_1m_1+\cdots+a_im_i \geqq \{n_1s_1^2+\cdots+n_is_i^2\}^{\frac{1}{2}} \tan\alpha$$

tends to

$$\left\{\int_\alpha^{\pi/2} \cos^{t-1}\phi d\phi\right\}\left\{\int_0^{\pi/2} \cos^{t-1}\phi d\phi\right\}^{-1},$$

where $t=n_1+\cdots+n_i-i$ and $0 \leqq \alpha < \pi/2$. The case $i=2$ is known and is used for the test that the samples are random and from the same normal population [cf. Yule-Kendall, Theory of Statistics, Griffin, London, 1937]. *W. Feller* (Providence, R. I.).

Wald, Abraham. On a statistical problem arising in the classification of an individual into one of two groups. Ann. Math. Statistics 15, 145–162 (1944). [MF 10822]

Let $x_{i\alpha}$ ($i=1, \cdots, p$; $\alpha=1, \cdots, N_1$) be a set of independent observations from a p-variate normal population Π_1 with means $\mu_1, \cdots, \mu_p$ and covariance matrix $\|\sigma_{ij}\|$, and let $y_{i\beta}$ ($i=1, \cdots, p$; $\beta=1, \cdots, N_2$) be a set of observations from a normal population Π_2 with means $\nu_1, \cdots, \nu_p$ and covariance matrix $\|\sigma_{ij}\|$. The problem is to test the hypothesis H_1 that the single p-variate observation $z_1, \cdots, z_p$, drawn from either Π_1 or Π_2, actually came from Π_1.

On the basis of an analogy to a procedure which, if $\mu_1, \cdots, \mu_p, \nu_1, \cdots, \nu_p$ and $\|\sigma_{ij}\|$ were known, would be the most powerful according to a result of Neyman and Pearson, the author proposes the statistic

$$U = \sum_i \sum_j S^{ij} z_i(\bar{y}_j - \bar{x}_j),$$

where

$$N_1\bar{x}_j = \sum_\alpha x_{j\alpha}, \quad N_2\bar{y}_j = \sum_\beta y_{j\beta},$$

$$(N_1+N_2-2)S_{ij} = \sum_{\alpha=1}^{N_1}(x_{i\alpha}-\bar{x}_i)(x_{j\alpha}-\bar{x}_j)+\sum_{\beta=1}^{N_2}(y_{i\beta}-\bar{y}_i)(y_{j\beta}-\bar{y}_j)$$

and $\|S^{ij}\|$ is the inverse of the matrix $\|S_{ij}\|$. The critical region consists of the large values of U. When N_1 and N_2 are large the distribution of U is approximately normal. In that case the approximate power is given, and the author uses this to give a procedure for a reasonable determination of the size of the critical region.

Determination of the exact distribution of U is a problem of formidable proportions. The author gives a number of results towards its solution. Among these is one which states that even for moderately large N_1+N_2 the distribution of a statistic V is well approximated by the distribution of $(N_1+N_2-2)m_3$, where m_1, m_2 and m_3 are three statistics whose joint distribution is obtained by the author. The distribution of U under both the null hypothesis and its alternative is a special case of the distribution of V.

J. Wolfowitz (New York, N. Y.).

Bowker, Albert H. Note on consistency of a proposed test for the problem of two samples. Ann. Math. Statistics 15, 98–101 (1944). [MF 10244]

A test of a hypothesis is said to be consistent if the probability of rejecting the hypothesis when some admissible alternative is true converges to one as the sample number approaches infinity. It is shown in this note that a test of the hypothesis that two samples are from the same population, recently proposed by H. C. Mathisen, is not consistent if for any arbitrary pair $(F(x), G(x))$ of distribution functions the hypothesis that the distribution of the first population is $F(x)$ and that of the second population is $G(x)$ is an admissible alternative to the hypothesis under test. However, as is pointed out by the author, Mathisen's test will be consistent if some proper restrictions are imposed on the set of admissible alternative hypotheses. For example, if the admissible alternatives are restricted to those where $G(x)=F(x+c)$, c a constant, Mathisen's test is consistent.

A. Wald (New York, N. Y.).

Bancroft, T. A. On biases in estimation due to the use of preliminary tests of significance. Ann. Math. Statistics 15, 190–204 (1944). [MF 10826]

The quantities s_1^2, s_2^2 are estimates of variance, where $n_1s_1^2/\sigma_1^2$, $n_2s_2^2/\sigma_2^2$ are independent and follow the χ^2 distribution with n_1 and n_2 degrees of freedom, respectively. If $\sigma_2^2 \leqq \sigma_1^2$ and it is desired to estimate σ_1^2, a procedure sometimes used in applied statistics is to make a preliminary z or F-test of the ratio s_1^2/s_2^2. If this is nonsignificant at a selected significance level, $(n_1s_1^2+n_2s_2^2)/(n_1+n_2)$ is used to estimate σ_1^2; if s_1^2/s_2^2 is significant, s_1^2 is used. The author calculates the bias and the sampling variance given by this procedure as functions of n_1, n_2, σ_1^2/σ_2^2 and the significance level. Two numerical examples are discussed.

Given the linear regression $Y=b_1x_1+b_2x_2$ of a variate y on variates x_1, x_2 as estimated from a sample, it is desired to estimate the population regression coefficient β_1 of y on x_1. If b_2 is not significant by a t-test the variate x_2 is sometimes dropped, the estimate b_1' from the regression of y on x_1 alone being used; on the other hand, if b_2 is significant, b_1 is used. The bias in the estimate of b_1 by this method is obtained as a function of β_2 (the population regression coefficient of y on x_2), ρ (the correlation coefficient between x_2 and x_1) and the significance level employed in the t-test of b_2. The bias does not depend on the value of β_1; it vanishes if ρ or β_2 is zero and is positive when ρ and β_2 are of like sign. Some numerical examples are discussed.

W. G. Cochran (Princeton, N. J.).

Hartley, H. O. Studentization or the elimination of the standard deviation of the parent population from the random sample-distribution of statistics. Biometrika 33, 173–180 (1944). [MF 10898]

Let $x_1, \cdots, x_m$ be a sample from a normal population with unknown variance σ^2. The "Studentized" statistics r considered are quotients $r=W/S$, where W is a statistic homogeneous of the first degree in the x's, S^2/σ^2 has the χ^2-distribution with n degrees of freedom and W and S are statistically independent. Approximate formulas suitable for the numerical tabulation of the distribution of r are obtained. The development is based on a recurrence formula in n, obtained previously [Suppl. J. Roy. Statist. Soc. 5, 80–88 (1938)]. From this is derived a partial differential equation, which is then solved by successive approximation. Use of the approximate formulas is illustrated by an application to the incomplete beta-function.

H. Scheffé.

Geary, R. C. Comparison of the concepts of efficiency and closeness for consistent estimates of a parameter. Biometrika 33, 123–128 (1944). [MF 10895]

Suppose a distribution depends in a known way on k parameters $\theta_1, \cdots, \theta_k$. Let X and Y be statistics calculated from a sample of size n. Pitman [Proc. Cambridge Philos. Soc. 33, 212–222 (1937)] has proposed the definition that X is a "closer" estimate of θ_1 than Y if

$$\text{Prob}\,\{|X-\theta_1|<|Y-\theta_1|\}>\tfrac{1}{2}$$

for all permissible values of $\theta_1, \cdots, \theta_k$. He pointed out that the relationship "closer than" is not transitive. Geary proves that, if X and Y are jointly normally distributed, each with mean θ_1, for all θ_i, then X is a closer estimate of θ_1 than Y if and only if $\text{Var}(X)<\text{Var}(Y)$ for all θ_i. This permits some comparison of R. A. Fisher's "efficiency" of estimates with Pitman's "closeness." Since the former is merely an asymptotic property, and the latter as defined by Pitman is not, perhaps "asymptotic closeness" should be defined and compared with "efficiency." Most of the paper is devoted to interesting particular cases, some of which involve estimates not asymptotically normal.

H. Scheffé (Syracuse, N. Y.).

Geary, R. C. Relations between statistics: the general and the sampling problem when the samples are large. Proc. Roy. Irish Acad. Sect. A. 49, 177–196 (1944). [MF 9985]

Let $X_1, \cdots, X_k$ be a set of k variates each of which is assumed to be the sum of two variates x and ξ, that is, $X_i=x_i+\xi_i$. It is assumed, furthermore, that the variates ξ_i are independent of one another and of the x_i. Suppose that only the values of the variates X_i ($i=1, \cdots, k$) are observed but not those of x_i and ξ_i, and that the variates x_i satisfy some functional relationships. The problem considered by the author is that of finding these functional relationships on the basis of the observations on the variates X_i. A general method for dealing with problems of this type, based on the fact that for indefinitely large samples certain semi-invariants of the X_i are equal to the corresponding semi-invariants of the x_i, has been given by the author in a previous paper [Proc. Roy. Irish Acad. Sect. A. 47, 63–76 (1942); these Rev. 4, 21]. In the present paper only linear relationships are considered and some simplifications and extensions of the earlier theory are given. The sampling problem is briefly discussed and approximations to the confidence limits of the unknown coefficients in the linear relationships are derived.

A. Wald (New York, N. Y.).

George, Aleyamma. On the problem of interval estimation. Sankhyā 6, 111–120 (1942). [MF 8633]

It is well known that the confidence coefficient alone does not uniquely determine confidence intervals. The author tries to compare the effect of various additional conditions

on confidence intervals for the parameters of the Pearson type III and the normal distribution. It is possible to give to some of her results a correct but trivial meaning.

J. Wolfowitz (New York, N. Y.).

Radhakrishna Rao, C. On balancing parameters. Science and Culture **9**, 554–555 (1944). [MF 10964]

Nair, K. R. The recovery of inter-block information in incomplete block designs. Sankhyā **6**, 383–390 (1944). [MF 10619]

In incomplete block experiments the t treatments are arranged in groups (called blocks) of size k, where $t>k$. A consequence is that comparisons among the total yields of the different blocks contain information about the effects of the treatments. Methods for utilizing this inter-block information in the statistical analysis of the results of the experiment were developed by Yates [Ann. Eugenics **9**, 136–156 (1939)] for balanced incomplete block designs and for the simpler types of partially balanced designs. Nair extends the procedure to the more general type of partially balanced design developed by Bose and Nair [Sankhyā **4**, 337–372 (1939)] and by Kishen [Sankhyā **5**, 329–344 (1941); these Rev. **4**, 108]; he shows how to complete the analysis of variance and estimate the standard errors of the treatment differences. The application of the method to the case where part of the data are missing is discussed briefly.

W. G. Cochran (Princeton, N. J.).

Hurwicz, Leonid. Stochastic models of economic fluctuations. Econometrica **12**, 114–124 (1944). [MF 10560]

The author shows that every system of linear stochastic difference equations of order h in N variables may be transformed into a first order system in Nh variables. On the other hand, it is shown that every system of first order in N variables may be transformed into a system of at most Nth order in one variable. Let x_t be this variable and $x_t+a_1x_{t-1}+\cdots+a_Nx_{t-N}=\eta_t$, where η_t is a random variable, be the equation satisfied by x_t. The random variable η_t will in general be autocorrelated, that is to say, $E(\eta_t\eta_{t'})$ will in general be different from 0 if $t\neq t'$. If the random terms in the original first order system are not autocorrelated, then $E(\eta_t\eta_{t'})=0$ for $t-t'\geqq N$ and $r_k=E(x_{t+k}x_t)$ satisfies, for $k\geqq N$, the equation $r_k+a_1r_{k-1}+\cdots+a_Nr_{k-N}=0$. The author also gives several instructive examples.

H. B. Mann (Columbus, Ohio).

Shephard, Ronald W. A mathematical theory of the incidence of taxation. Econometrica **12**, 1–18 (1944). [MF 10070]

The author considers a simplified mathematical model of the economy of the type introduced by G. C. Evans. In this model the entrepreneurs maximize their profits under conditions of strict competition. It is assumed that a state of equilibrium exists with no accumulations of stocks in any compartment of the economy. In this model a study is made of the incidence of the following kinds of taxes: (1) a tax on the value of capital used in production; (2) a tax on the money value of production; (3) a tax on the money value of wages to provide benefit payments for the unemployed. The author arrives at various conclusions depending on the assumptions made concerning the use of the tax revenue and the offer of labor services.

A. Wald.

Richter, Hans. Untersuchungen zum Erneuerungsproblem. Math. Ann. **118**, 145–194 (1941). [MF 10700]

The author studies the solution of the integral equation

$$\phi(t)=P(t)+\int_0^t P(t-\tau)\phi(\tau)d\tau,$$

and, in particular, the problem when $\phi(t)$ does or does not approach a finite limit as t becomes infinite. This integral equation has applications in renewal theory. [For a more elementary treatment of a very similar problem, see Feller, Ann. Math. Statistics **12**, 243–267 (1941); these Rev. **3**, 151.]

A. E. Heins (Cambridge, Mass.).

Schwarz, Hans. Zur "wahrscheinlichkeitstheoretischen Stabilisierung" beim Erneuerungsproblem. Math. Ann. **118**, 771–779 (1943). [MF 10725]

The author refines some of the analysis put forth by Richter in the paper reviewed above.

A. E. Heins.

Mathematical Biology

Geiringer, Hilda. On the probability theory of linkage in Mendelian heredity. Ann. Math. Statistics **15**, 25–57 (1944). [MF 10237]

The principal results obtained are the following. (1) For the description of linkage phenomena among m genes, a probability distribution containing 2^{m-1} disposable values is required. Two such distributions are obtained, and their relations to each other and to the biologist's crossover probabilities are discussed. (2) Under the hypotheses of (a) random mating, (b) equal survival values, (c) absence of mutation and (d) identical crossover relations in the two sexes, it is shown that any set of genes reaches a limiting distribution, and that, in the absence of complete linkage, this distribution is one in which genes are combined at random. (3) The linear theory of the ordering of the genes is considered. Distance between two genes is defined as the expectation of the number of crossovers between these genes. [In actual applications, the author seems to assume that double crossovers cannot occur between next neighboring genes.] There seems to be no difference between this definition and that used by Morgan. (4) Suggestions are made looking towards crossover distribution models, with particular reference to the phenomenon of interference. It does not appear, however, that the models actually proposed correspond to the biological facts.

C. P. Winsor.

Mittmann, Orfrid. Funktionale Zusammenhänge in erbbiologischen Gesamtheiten. Arch. Math. Wirtsch.-Sozialforschg. **6**, 70–80 (1940). [MF 10649]

Identical with a paper under another title [Deutsche Math. **5**, 563–570 (1941); these Rev. **3**, 10].

W. Feller.

Geppert, Maria-Pia. Über die Alterskorrektur von Merkmalshäufigkeiten in der Erbstatistik. Arch. Math. Wirtsch.-Sozialforschg. **6**, 80–102 (1940). [MF 10650]

Malécot, Gustave. Mendélisme et consanguinité. C. R. Acad. Sci. Paris **215**, 313–314 (1942). [MF 9502]

McCulloch, Warren S. and Pitts, Walter. A logical calculus of the ideas immanent in nervous activity. Bull. Math. Biophys. 5, 115–133 (1943). [MF 10773]

The all or none law for neural activity is utilized to set up a calculus for nervous nets based on symbolic logic. A number of theorems are proved involving conditions for the existence of a net with certain properties, equivalence of different assumptions regarding neural behavior, equivalence of different nets, etc. The paper concludes with an application to certain psychological and philosophical problems. *C. E. Shannon* (New York, N. Y.).

Landahl, H. D., McCulloch, W. S. and Pitts, Walter. A statistical consequence of the logical calculus of nervous nets. Bull. Math. Biophys. 5, 135–137 (1943). [MF 10774]

A theorem is proved which enables one to determine the mean frequency of nervous impulses on a neuron from the logical expression for the network [see the preceding review]. The method is applicable to nets with no circular paths, and to some other cases, and requires the substitution of numerical operations and values for logical ones.
C. E. Shannon (New York, N. Y.).

Lettvin, Jerome Y. and Pitts, Walter. A mathematical theory of the affective psychoses. Bull. Math. Biophys. 5, 139–148 (1943). [MF 10775]

A theory of the affective psychoses is developed in which two variables, the intensity of emotion $\varphi(t)$ and of activity $\psi(t)$, are assumed to vary with time in accordance with a pair of integrodifferential equations of the dynamical type. These are deduced on the assumption that the change in φ or ψ in a small interval of time is linear in the present environment, the partially forgotten effects of the past, and in a homeostatic restoring force. The solutions can be represented as trajectories in a φ, ψ plane and questions of stable points are investigated. *C. E. Shannon.*

Wilson, Edwin B. and Worcester, Jane. A second approximation to Soper's epidemic curve. Proc. Nat. Acad. Sci. U. S. A. 30, 37–44 (1944). [MF 10114]

If A is the rate at which new susceptibles come into the population, S the number of susceptibles at any time, the authors put $u=\log(A-dS/dT)-\log m$, where m is the constant value of S under the assumption of a steady state and the time T is measured in suitable units. The authors prove that the exact equation of Soper's theory is

$$u'(T)-u'(T+1)=\exp u(T)-(A/m)\exp[u(T)-u(T+1)],$$

where $u'(T)=du/dT$. They show the mathematical meaning of Soper's approximation as well as an inconsistency to which it leads. Several better approximations are proposed in the case in which the epidemic is so short that one may take $A\approx 0$. A numerical analysis of three of these improved methods gives an epidemic curve which is considerably different from Soper's approximation. *I. Opatowski.*

García, Godofredo. On the integration of the complete equation of diffusion and its application to the study of cells. Revista Ci., Lima 46, 293–325 (1944). (Spanish) [MF 10983]

The author rederives the diffusion equation, with convection, for a metabolizing cell; he then proposes to consider first the case of a parallelepipedal cell, without convection, with autocatalytic metabolic reaction. A particular solution is exhibited of the form

$$e^{\delta t}(\cos\alpha x\cos\beta y\cos\gamma z+\sin\alpha x\sin\beta y\sin\gamma z),$$

after which fifty-two numbered equations are written with no explicit statement of boundary conditions [evidently, however, the concentration is prescribed at the boundary, which is a case of no great biophysical interest] nor any other explanation. Passing to the "complete equation of diffusion" the author assumes small convection with a velocity potential and subjects the equation of continuity to certain transformations. Again undefined symbols are introduced and in general this part of the paper is scarcely more intelligible than the other. *A. S. Householder.*

GEOMETRY

Birkhoff, Garrett. Metric foundations of geometry. I. Trans. Amer. Math. Soc. 55, 465–492 (1944). [MF 10471]

Let $\mathfrak{S}$ denote the class of all finite dimensional Euclidean, hyperbolic and spherical spaces. The author's object is to show that the class $\mathfrak{S}$ is characterized among the class of all metric spaces by the following properties. (1) Through any two points a segment may be drawn, locally unique. (2) Any isometry between subsets of space can be extended to a self-isometry of all space. These properties are similar to those used by Busemann for the same purpose [Amer. J. Math. 63, 101–111 (1941); these Rev. 2, 258; see also Metric Methods in Finsler Spaces, Princeton University Press, Princeton, N. J., 1942, chap. 5, section 2; these Rev. 4, 109]. The author prefers to assume property (1) as primitive rather than deriving it from basic metric properties. (This preference precludes any contribution to the problem of characterizing subsets of the spaces of $\mathfrak{S}$, for which property (1) is far too restrictive.) The strength of property (2) (free mobility) is effectively exploited by the author to establish easily some of the most important theorems. The validity of property (2) for infinite sets insures the finite dimensionality of the space. On the other hand, it casts out elliptic space, which contains congruent triples for which the property fails. The methods are elementary, direct geometric arguments being used. The proof of the congruence with a member of $\mathfrak{S}$ is postponed to a later paper. It is observed that the definition of betweenness given on page 468 is defective in that it omits the essential requirement that the point x be distinct from point a and from point b.
L. M. Blumenthal (Columbia, Mo.).

Blumenthal, Leonard M. Remarks on a weak four-point property. Revista Ci., Lima 45, 183–193 (1943). [MF 10046]

A metric space satisfies the tetrahedral inequality if, for each quadruple of points p,q,r,s,

$$D(p,q,r,s)=\begin{vmatrix}0&1&1&1&1\\1&0&(pq)^2&(pr)^2&(ps)^2\\1&(pq)^2&0&(qr)^2&(qs)^2\\1&(pr)^2&(qr)^2&0&(rs)^2\\1&(ps)^2&(qs)^2&(rs)^2&0\end{vmatrix}\geqq 0.$$

A metric space is Ptolemaic if $C(p,q,r,s)\leqq 0$ for every point quadruple, where C is the minor of the first element of the principal diagonal of D. A metric space satisfying the tetrahedral inequality is Ptolemaic, but the converse is not

true. A metric Ptolemaic space satisfies all the Huntington and Kline postulates for linear betweenness except that one which states that in a pair-wise distinct triple of points one is between the other two. *J. L. Dorroh.*

Schmidt, Arnold. Die Dualität von Inzidenz und Senkrechtstehen in der absoluten Geometrie. Math. Ann. **118**, 609–625 (1943). [MF 10717]

In many of the earlier investigations in the foundations of geometries which were stimulated by the work of Hjelmslev (for example, von Thomsen, Podehl and Reidemeister, Bachman), the notion of "reflection" played a fundamental if not primitive role. The writer, in the present paper, fully justifies this by making "reflection" (in a line and in a point) a primitive notion. His system of axioms is remarkable for exhibiting a duality between incidence and perpendicularity. *L. M. Blumenthal* (Columbia, Mo.).

Barbilian, D. Aufbau der projektiven Geometrie in der absoluten Ebene. Monatsh. Math. Phys. **50**, 298–316 (1943). [MF 10483]

The construction of a model for a given geometry in terms of the elements of another geometry is always tempting. Here the author builds projective geometry using as elements the horocycles of hyperbolic geometry. The chief problem is to adjust the irrationalities involved, and to do this he introduces two "orientations." If the equation of the absolute conic in the hyperbolic plane is taken to be $x^2+y^2-1=0$, then the equation of a horocycle may be written

$$(h+k)(x^2+y^2)-2ka_1x-2ka_2y-h+k=0,$$

where

$$a_1=\frac{\lambda^2-\mu^2}{\lambda^2+\mu^2},\quad a_2=\frac{2\lambda\mu}{\lambda^2+\mu^2}.$$

The first orientation is based on the sign attached to the radius $r^\epsilon=\epsilon h/(h+k)$, $\epsilon=\pm1$, and the second on the sign of the radical $\rho^{\epsilon,\eta}=\eta(hk(\lambda^2+\mu^2))^{\frac{1}{2}}$, $\eta=\pm1$. The correspondence is based on the association $x_0^-=\rho^{-,\eta}$, $x_1^-=k^-\lambda^-$, $x_2^-=k^-\mu^-$ and $x_0^+=\rho^{+,\eta}$, $x_1^+=k^+\mu^+$, $x_2^+=-k^+\lambda^+$. The homogeneous coordinates x_i^- represent a point of the projective plane, x_i^+ a line, and these may be normalized so that the condition of incidence becomes $x_i^-x_i^+=0$, which corresponds to the contact of the corresponding horocycles. Theorems of projective geometry may be translated into theorems of hyperbolic geometry; in particular, Desargues' theorem is equivalent to Bricard's version of Feuerbach's theorem. The extension to n dimensions is given as is also that to Euclidean geometry. *G. de B. Robinson.*

Gerretsen, J. C. H. Die Begründung der Trigonometrie in der hyperbolischen Ebene. I. Nederl. Akad. Wetensch., Proc. **45**, 360–366 (1942). [MF 10403]

In this series of three papers [cf. the following reviews] the author works through a suggestion made by Hilbert for the foundation of trigonometry in hyperbolic geometry. The procedure is straightforward and is based on the notion of reflection in a line as determining the operations of addition and multiplication of points on the absolute conic Σ.

Choose first an arbitrary line whose "points at infinity" or "ends" on Σ are denoted by 0 and ∞. Denoting a reflection in a line a by $\mathfrak{S}_a$, then, if three lines a, b, c are parallel, having the same end ω, there exists a line d through ω such that $\mathfrak{S}_c\mathfrak{S}_b\mathfrak{S}_a=\mathfrak{S}_d$. In particular, if reflection in the line (α, ∞) is denoted $\mathfrak{S}_\alpha$, then $\mathfrak{S}_\beta\mathfrak{S}_0\mathfrak{S}_\alpha=\mathfrak{S}_\sigma$ and σ is defined to be the sum $\alpha+\beta$ of the ends α, β on Σ; addition is commutative and associative and $\mathfrak{S}_0\mathfrak{S}_0\mathfrak{S}_\alpha=\mathfrak{S}_\alpha$. Also, the ends of any line perpendicular to $(0, \infty)$ are a and $-a$. Choosing a point O on $(0, \infty)$ and denoting the ends of the line through O perpendicular to $(0, \infty)$ by $+1$ and -1, it is possible to define the multiplication of ends. This multiplication is commutative, associative and distributive and the inverse of an end α is obtained by reflection in the line $(1, -1)$. *G. de B. Robinson* (Ottawa, Ont.).

Gerretsen, J. C. H. Die Begründung der Trigonometrie in der hyperbolischen Ebene. II. Nederl. Akad. Wetensch., Proc. **45**, 479–483 (1942). [MF 10413]

[Cf. the preceding and the two following reviews.] After giving the standard decomposition of a bilinear transformation as applied to the points of Σ, the author shows that a necessary and sufficient condition for a line (α, α') to intersect $(0, \infty)$ in a point P is that $\alpha\alpha'=-\pi^2$, where $(\pi, -\pi)$ is perpendicular to $(0, \infty)$ at P.

Assuming Hilbert's axioms of congruence, a metric is introduced in the following manner. Let (α, α') be the line through A and (β, β') that through B perpendicular to the segment AB; denote the cross-ratio $\{\alpha\alpha', \beta\beta'\}$ by δ. In particular, if A lies on $(0, \infty)$, the cross-ratio associated with the segment OA of congruence class a is

$$\delta=((\alpha-1)/(\alpha+1))^2,$$

and we write $\exp a=\alpha=(1+\sqrt{\delta})/(1-\sqrt{\delta})$. The addition theorem for the exponential function follows without difficulty, making it possible to define the hyperbolic functions of a and conclude that $\tanh \frac{1}{2}a=\sqrt{\delta}$, for $a>0$.
G. de B. Robinson (Ottawa, Ont.).

Gerretsen, J. C. H. Die Begründung der Trigonometrie in der hyperbolischen Ebene. III. Nederl. Akad. Wetensch., Proc. **45**, 559–566 (1942). [MF 10421]

[Cf. the two preceding reviews and the following review.] The introduction of trigonometry into the hyperbolic geometry follows similar lines. Associate with the angle $\angle APB$, where PA is the line (α, α') and PB the line (β, β'), the cross-ratio $\delta^*=\{\alpha\alpha', \beta\beta'\}$. In particular, for the angle A defined by the lines $(0, 1)$ and $(0, \alpha)$,

$$\delta^*=-((\alpha-1)/(\alpha+1))^2,$$

so that δ^* is essentially negative and we write $\tan \frac{1}{2}A=(-\delta^*)^{\frac{1}{2}}$. As before, the addition theorem for the tangent along with the usual formulae of elementary trigonometry follow directly. The deduction of the important relation connecting the length of a segment x and the angle of parallelism $\Pi(x)$ is immediate. The formulae for a right angled triangle are obtained in detail.

In conclusion the author gives trigonometric proofs that a line which contains at least one point within (and a circle which contains at least one point within), and one point without, a given circle intersects this circle in two points.
G. de B. Robinson (Ottawa, Ont.).

Gerretsen, J. C. H. Zur hyperbolische Geometrie. Nederl. Akad. Wetensch., Proc. **45**, 567–573 (1942). [MF 10422]

This paper is a sequel to the three papers reviewed above. In it the author gives a construction for associating two points ξ, η of Σ with each point P of the plane, and this association is (1, 1). Taking ξ, η as Cartesian coordinates, the lines of the plane go into the lines and circles orthogonal to the fixed line $\eta=0$, and we have Poincaré's model of hyperbolic geometry. *G. de B. Robinson.*

Mann, Henry B. On orthogonal Latin squares. Bull. Amer. Math. Soc. **50**, 249–257 (1944). [MF 10207]

This provides a simple proof for the theorem of C. R. MacInnes [Amer. Math. Monthly **14**, 171–174 (1907)] that $PG(2, 5)$ is the only projective plane geometry with six points on a line. It does not differ essentially from the treatment of such theorems by R. C. Bose and K. R. Nair [Sankhyā **5**, 361–382 (1941); these Rev. **4**, 33].

H. S. M. Coxeter (Toronto, Ont.).

✠de Oliveira Júnior, Ernesto Luiz. The projective coordinates in the forms of first, second and third kind. Imprensa Nacional, Rio de Janeiro, 1943. 62 pp. (Portuguese) [MF 10593]

This is an expository article dealing with the establishment of homogeneous projective coordinate systems in fundamental forms of one, two and three dimensions and giving a brief development of the analytic projective geometry of these forms. The author chooses as the basis of a system of coordinates in a one-dimensional form the notion of an harmonic (Möbius) scale. If, for example, P is a parabolic projectivity defined on a point range and A_0 is any point of the range distinct from the single self-corresponding point M, the points $\cdots A_{-n}\cdots A_{-2}A_{-1}A_0A_1A_2\cdots A_n\cdots M$, where $A_{\pm i}$ arises from repeated application of P and its inverse to A_0, form an harmonic scale (each A_i is the harmonic conjugate of M with respect to A_{i-1}, A_{i+1}). After the introduction of coordinates in one-dimensional forms, systems in two and three dimensions are established in a conventional manner [e.g., Graustein, Introduction to Higher Geometry, Macmillan, New York, 1930, pp. 156–157, 419–420].

L. M. Blumenthal (Columbia, Mo.).

Daus, P. H. Correlations in terms of central collineations and central correlations. Univ. California Publ. Math. (N.S.) **2** [No. 1, Seminar Rep. in Math. (Los Angeles)], 63–75 (1944). [MF 10455]

A correlation in a planar field is uniquely determined by association of a complete quadrangle with a complete quadrilateral. The author's aim is to establish such a correspondence by a chain of geometric constructions, "the ultimate aim being a systematic development and classification of correlations from this two-dimensional geometric viewpoint."

L. M. Blumenthal (Columbia, Mo.).

Lasley, J. W., Jr. On the classification of collineations in the plane. Nat. Math. Mag. **19**, 11–20 (1944). [MF 11151]

Hohenberg, Fritz. Apolarität und Schliessungsproblem bei Kegelschnitten. Monatsh. Math. Phys. **50**, 111–124 (1941). [MF 10475]

Problems in apolarity and certain properties of closure are considered in connection with the system of conics of a plane having a common self-polar triangle. The sides of this triangle are taken as the triangle of reference. Each conic of the system having the reference triangle as self-polar is then mapped as a point on an auxiliary plane, the coordinates of which are the coefficients of the equation of the conic. Finally, this system (P) and the system of conics (O) are superposed. Many of the properties of such systems of conics having a common self-polar triangle were considered by K. Meister [Z. Math. Phys. **31**, 321–347 (1886)] but the mapping was not used.

By means of this mapping, the line elements of a line in (P) are mapped on the line elements of a line in (O). It is a contact transformation in the sense of S. Lie. A conic (α) is said to be harmonically circumscribed to $(\bar{\alpha})$ if there exists a polar triangle of $(\bar{\alpha})$ with vertices on (α). The system forms a pencil. If (α) is harmonically inscribed to $(\bar{\alpha})$, then $(\bar{\alpha})$ is harmonically circumscribed to (α). Such systems are called apolar. This method of approach to apolarity is different from the usual one. The property is invariant under all collineations that leave the conic (α) invariant. The equianharmonic conics of the system play a double role. The method is then applied to problems in closure. If one triangle exists, circumscribed to (α), then an infinite number of such triangles exist (poristic property). The conics inscribed in (α) envelope a (projective) asteroid in $(\bar{\alpha})$. If a conic of the system is taken as the absolute of a non-Euclidean metric, various quasi-metric properties result. These are applied to some systems.

V. Snyder.

McBrien, V. O. Cardioids associated with a cyclic quadrangle. Amer. Math. Monthly **51**, 74–77 (1944). [MF 10066]

The envelope of the Kantor lines of a pencil of lines on a fixed point z_0 with respect to a cyclic quandrangle is a deltoid (three-cusped hypocycloid). The locus of the orthopoles of the pencil with respect to the four triangles determined by the vertices of the quadrangle is four ellipses. Each of these ellipses is tangent to the deltoid in at least one point. Four cardioids are associated with the quadrangle in the determination of the number of points of contact of the ellipses with the deltoid as a function of the position of z_0.

J. L. Dorroh (Baton Rouge, La.).

Frame, J. S. Tangent triangles to a biquadratic curve. Amer. Math. Monthly **51**, 445–450 (1944). [MF 11245]

Venkataraman, M. and Kesava Menon, P. On B. R. Venkataraman's chain of theorems in circle geometry. Math. Student **11**, 31–32 (1943). [MF 10996]

Konnully, Augustine O. Orthocentre of a cyclic polygon. Math. Student **11**, 28–30 (1943). [MF 10995]

Thébault, Victor. Sur les sphères de Tücker du tétraèdre. C. R. Acad. Sci. Paris **217**, 257–259 (1943). [MF 11110]

Bouvaist, Robert et Thébault, Victor. Sur les triangles isopolaires. C. R. Acad. Sci. Paris **217**, 223–225 (1943). [MF 10645]

Krames, Josef. Über die mehrdeutigen Orientierungen zweier Sehstrahlbündel und einige Eigenschaften der orthogonalen Regelflächen zweiten Grades. Monatsh. Math. Phys. **50**, 65–83 (1941). [MF 10493]

Two perspectives of a surface, together with their interior orientations, determine two bundles of rays. These bundles are said to be oriented if corresponding rays intersect. Corresponding rays are rays connecting either center with the image of the same object point. The problem of photogrammetric reconstruction is then the problem of bringing the two bundles into an oriented position. This problem has generally a unique solution; multiple solutions occur only if the surface surveyed is an orthogonal quadric surface [Monatsh. Math. Phys. **49**, 327–354 (1941); these Rev. **3**, 300]. In this paper the author considers the transition from one oriented position to another. This transition can be made by rotating each bundle about one of its lines. Corresponding axes of rotation form four pairs of projective pencils of lines.

E. Lukacs (Berea, Ky.).

Krames, Josef. Über bemerkenswerte Sonderfälle des "Gefährlichen Ortes" der photogrammetrischen Hauptaufgabe. Monatsh. Math. Phys. **50**, 1–13 (1941). [MF 10487]

In a previous paper [Monatsh. Math. Phys. **49**, 327–354 (1941); these Rev. **3**, 300] the author showed that under certain conditions the problem of photogrammetric reconstruction (Hauptaufgabe der Photogrammetrie) has not a unique solution. In this paper he discusses a special case where the three solutions are congruent. *E. Lukacs.*

Krames, Josef. Der einfachste Übergang zur Nebenlösung bei vorliegendem "Gefährlichen Ort." Monatsh. Math. Phys. **50**, 84–100 (1941). [MF 10494]

Pursuing his investigations on the problem of photogrammetric reconstruction the author discusses properties of orthogonal ruled quadric surfaces. Statements of his previous papers [Monatsh. Math. Phys. **49**, 327–354 (1941); these Rev. **3**, 300; see also the two preceding reviews] are again derived and the quadratic correspondence between multiple solutions of the problem of photogrammetric reconstruction is studied. *E. Lukacs* (Berea, Ky.).

Wunderlich, Walter. Zur Eindeutigkeitsfrage der Hauptaufgabe der Photogrammetrie. Monatsh. Math. Phys. **50**, 151–164 (1941). [MF 10479]

The author discusses the case where the problem of photogrammetric reconstruction has multiple solutions. Results already given by Krames [cf. the three preceding reviews] are restated and the proofs slightly modified.

E. Lukacs (Berea, Ky.).

Dubuisson, Bernard. Sur les applications à l'aérotechnique du redressement des photographies aériennes. C. R. Acad. Sci. Paris **216**, 867–869 (1943). [MF 10659]

Herrera, Émile. Sur les cartes orthométriques à double projection. C. R. Acad. Sci. Paris **217**, 275–276 (1943). [MF 11113]

Herrick, Samuel. Grid navigation. Geographical Review **34**, 436–456 (1944). [MF 10925]

The paper describes a method for navigating a straight line on any conformal chart as readily as on a Mercator chart by means of the introduction of a rectangular grid and corresponding lines of constant grid variation.

O. Neugebauer (Providence, R. I.).

Convex Domains, Integral Geometry

Haupt, Otto. Bemerkungen über Konvexbogen. Monatsh. Math. Phys. **50**, 339–367 (1943). [MF 10486]

The author studies (1) the relative position of two open or closed convex curves and the arrangement of their points of intersection on them, (2) convergent sequences of such curves. The following generalization of a theorem by Boehmer and Mukhopadhyaya seems of interest. Given a closed curve C of second order in the projective plane (that is, C shall have at most two points in common with any real straight line), a straight line L and an arc A of second order. The intersections AL, CL, AC shall contain no, two, at least five points, respectively. Thus CL divides C into two partial arcs. Then all the points of AC lie on the same partial arc. [In the paper, this theorem is formulated slightly more generally.] *P. Scherk* (Saskatoon, Sask.).

Fejes, L. Über eine Extremaleigenschaft der Kegelschnittbogen. Monatsh. Math. Phys. **50**, 317–326 (1943). [MF 10484]

The author continues his own work and that of Lázár and Sas. He proves, among others, the following result. Let there be given a convex curve K of area A, and a polygon of n vertices P_n; $A(P_n, K)$ denotes the area of the points which are contained in one and only one of P_n and K. Then

$$A(P_n, K) < (\tfrac{1}{3}+\epsilon)(\pi^2/n^2)A$$

for large n and suitably chosen P_n. The constant $\frac{1}{3}$ is best possible, equality only for the ellipse. *P. Erdös.*

Parker, W. V. and Pryor, J. E. Polygons of greatest area inscribed in an ellipse. Amer. Math. Monthly **51**, 205–209 (1944). [MF 10256]

Kollros, Louis. Démonstrations de formules de Steiner. Comment. Math. Helv. **16**, 60–64 (1944).

Steiner announced two formulas for the area of an ellipse whose center O is given and which is inscribed in, or circumscribed to, a given triangle Δ [Steiner, Gesammelte Werke, Reimer, Berlin, 1882, vol. 2, p. 329]. The author adds to them a third which gives the area of an ellipse with center O for which Δ is a polar triangle, and furnishes two sets of proofs for the three formulas. The first set is particularly simple; after expressing all the lengths by means of areas, the general case is reduced by means of affinities to the case of a circle. Many applications are given.

P. Scherk (Saskatoon, Sask.).

Haupt, Otto. Über den Ovalsatz von Böhmer-Mukhopadhyaya. Math. Ann. **118**, 629–635 (1943). [MF 10719]

The numerical eccentricity of the conic $a^{-2}x^2 \pm b^{-2}y^2 = 1$ is the number $e = (1 \mp b^2a^{-2})^{\frac{1}{2}} \geqq 0$, where $0 < b \leqq a$ in the case of the ellipse. Thus $e > 1$ for the hyperbola and $e < 1$ for the ellipse. For the parabola we put $e = 1$. An oval $\mathfrak{C}$ is a Jordan curve which has at most two points in common with any straight line. A point $S \subset \mathfrak{C}$ is called sextactic if every neighborhood of S on $\mathfrak{C}$ contains at least six different points that lie on a conic. A point $P \subset \mathfrak{C}$ is a $(<e)$-point if the numerical eccentricity of the conic through any five points of $\mathfrak{C}$ that lie sufficiently close to P is smaller than e. A (<1)-point is called elliptic.

Generalizing a theorem by Boehmer, Mukhopadhyaya proved that, if $\mathfrak{C}$ has a unique tangent everywhere and if all the sextactic points of $\mathfrak{C}$ are elliptic, then the conic through any five points of $\mathfrak{C}$ is an ellipse [Math. Z. **30**, 560–571 (1929)]. His indirect proof starts with a hyperbola $\mathfrak{H}$ which meets $\mathfrak{C}$ in at least five points, hence in at least six. He then constructs a sequence of hyperbolas that have (at least) six points in common with $\mathfrak{C}$ which converge to the same point $S \subset \mathfrak{C}$. Then S is a nonelliptic sextactic point. By replacing the construction of the sequence by a continuous deformation of $\mathfrak{H}$, Haupt can drop the differentiability assumption and prove the following generalization. Let $e \geqq 1$. If all the sextactic points of $\mathfrak{C}$ are $(<e)$-points, then the numerical eccentricity of the conic through any five points of $\mathfrak{C}$ is smaller than e. *P. Scherk* (Saskatoon, Sask.).

Santaló, Luis A. Properties of convex figures on a sphere. Math. Notae **4**, 11–40 (1944). (Spanish) [MF 10697]

A convex curve on the unit sphere S is a closed curve which intersects any great circle in at most two points. Such a curve lies entirely in a hemisphere. A (closed) domain C on S is called convex if it is bounded by a convex

curve and lies on a hemisphere. Notation: ρ is the radius of the in-circle of C, D the diameter of C (all the distances are measured on S), L the length of the curve which bounds C, E the angle of the smallest lune containing C; thus L, D, E, $2\rho \leqq \pi$. The following theorems are proved. (1) If ρ is given, E is maximal if (but not only if) C is an equilateral triangle. The C's for which the maximal E is reached are determined. (2) For a given $D<\pi/2$ ($D \geqq \pi/2$), L is maximal if and only if C is a spherical Reuleaux triangle (the equilateral triangle of altitude D). The first theorem yields a theorem of Robinson as a simple corollary [Bull. Amer. Math. Soc. **44**, 115–116 (1938)]. The duals of the above theorems are stated. The author also discusses the C's of constant width. *P. Scherk* (Saskatoon, Sask.).

Fiala, F. Le problème des isopérimètres dans les plans de Riemann à courbure de signe constant. Comment. Math. Helv. **15**, 249–264 (1943).

In another paper [Comment. Math. Helv. **13**, 293–346 (1941); these Rev. **3**, 301; cited as I] the author has established an isoperimetric inequality on 2-dimensional simply connected Riemann manifolds with positive curvature. He now takes up the existence problem. Let $\bar{A}(L)$ be the sup of the areas determined by (simple closed) curves of length L; this is finite for all L below a critical L^* ($\leqq \infty$) which depends on the manifold; assume $L<L^*$. Theorem: there exists at least one curve of length L for which the area is equal to $\bar{A}(L)$. The proof is based on two facts. (a) For every divergent (to infinity) sequence of curves F_n of length L the relation lim sup $A_n \leqq L^2/4\pi$ holds (A_n is the area of the interior of F_n); this was proved in I. (b) $\bar{A}(L)>L^2/4\pi$; this is proved by means of the properties of true circles as developed in I (a true circle is the set of all points at a given distance from a center). Then (a) and (b) imply that a maximizing sequence of curves cannot be divergent; a maximizing curve can then be constructed. For the case of negative curvature the situation is different. Radó and Beckenbach have proved that for any curve $L^2>4\pi A$. It is now shown that there is no maximizing curve under one of the following conditions: (a) the curvatura integra of the manifold is finite; (b) the curvature becomes arbitrarily small outside of sufficiently big domain. This is done by showing that geodesic circles, which are "sufficiently far away," have their area arbitrarily near to $L^2/4\pi$, and that consequently $4\pi\bar{A}(L)=L^2$. *H. Samelson.*

Maak, Wilhelm. Ergänzung und Berichtigung zur Abhandlung in Band 118, S. 299. Math. Ann. **119**, 162–164 (1943). [MF 10098]

If K and K' are two continuous curves, K fixed and K' rigidly movable, then N, the number of crossings of K and K', is a function of the position of K', which is determined by three parameters x, y, φ, the first two being translations of the plane containing K' and the third a rotation. In the paper cited in the title of this note [see these Rev. **5**, 10] a proof was given of the formula

$$\underset{(\text{all } x,\, y,\, \varphi \text{ space})}{\iiint} N(x, y, \varphi)\,dx\,dy\,d\varphi = 4LL',$$

where L, L' are the lengths of K, K', respectively. If K or K' is not rectifiable, the corresponding length is considered to be ∞.

In this note the author shows that in the rectifiable case the formula above also holds if N is replaced by $\bar{N}$, the number of points in the intersection of K and K', the difference between N and $\bar{N}$ corresponding to the places where K and K' touch. This is proved by decomposing the curves into parts of equal length, forming the convex cover of each part and using an integral formula of Blaschke. If the nonrectifiable case, the author points out that his previous proof had a fault in it due to the fact that a certain function was defined as 1 or 0 according as a line segment had its two end points on opposite sides of a closed curve or both on the same side of a closed curve, nothing being said about the possibility that an end point may be on the closed curve. The proof is corrected in this note, except the weaker statement $\iiint \bar{N}\,dx\,dy\,d\varphi$ is obtained. *J. W. Green.*

Segre, B. and Mahler, K. On the densest packing of circles. Amer. Math. Monthly **51**, 261–270 (1944). [MF 10564]

Let Π be a convex polygon, no angle of which exceeds $120°$. (Then Π has at most 6 sides.) Let A be the area of Π. The authors prove that at most $A/\sqrt{12}$ nonoverlapping circles of radius 1 can be placed inside Π. The proof, though involved, uses only elementary geometrical considerations. It is based on the following lemma. Let Σ be a (finite or infinite) set of points in the plane, such that the distance of any two points of Σ is at least 2. Let P be a point of Σ. Denote by $S(P)$ the set of points in the plane which are closer to P then to any other point of Σ. Then the area $A(P)$ of $S(P)$ is at least $\sqrt{12}$. The equality $A=\sqrt{12}$ holds only if there are 6 points of Σ forming a regular hexagon of side 2 and center P. *F. John* (Aberdeen, Md.).

Hofreiter, N. Gitterförmige lückenlose Ausfüllung des R_n mit kongruenten Würfeln. Monatsh. Math. Phys. **50**, 48–64 (1941). [MF 10492]

This is an attempt to simplify the work of Perron on Minkowski's conjecture [Math. Ann. **117**, 415–447, 609–658 (1940–1941); these Rev. **2**, 153; **3**, 253]. The conjecture is again established for Euclidean space of not more than nine dimensions. Some of Perron's lemmas are used, along with the following. Given $2n$ numbers $\alpha_1, \cdots, \alpha_n; \beta_1, \cdots, \beta_n$ ($0 \leqq \alpha_i < 1$; $0 \leqq \beta_i < 1$), we can choose $\epsilon_i = \pm 1$ ($i=1, \cdots, n$) so that $|\sum \epsilon_i \alpha_i| < 1$, $|\sum \epsilon_i \beta_i| < 1$. *H. S. M. Coxeter.*

Algebraic Geometry

Montel, Paul. Sur la dispersion des points singuliers des courbes algébriques. Bull. Sci. Math. (2) **66**, 27–31 (1942). [MF 10463]

Some observations connected with a standard method (due to Bertini) of reducing the singularities of a plane curve by Cremona transformations. *R. J. Walker.*

Segre, B. A remark on unicursal curves lying on the general quartic surface. Quart. J. Math., Oxford Ser. **15**, 24–25 (1944). [MF 10685]

Segre, B. The maximum number of lines lying on a quartic surface. Quart. J. Math., Oxford Ser. **14**, 86–96 (1943). [MF 9935]

The maximum number of lines lying on a nonsingular algebraic surface of order four has often been discussed [W. Fr. Meyer, Enzyklopädie Math. Wiss. B. 3, Teil 2, Hälfte 2, pp. 1533–1779, §54] but not solved. Surfaces with 64 lines are known from the examples given by F.

Schur. The present paper proves that the maximum is 64. The equation of an algebraic surface of order n containing the given line and the pencil of residual plane curves through it is first considered. Every such curve intersects the given line in $n-1$ points which are inflexions on the plane curve or the points on the line, each of which is an inflexion, are $8n-14$ in number. The consequences of each alternative are examined in detail. Applied to the case $n=4$, the theorem follows. *V. Snyder* (Ithaca, N. Y.).

Green, H. Gwynedd. The focal form of the quadric in n dimensions. Proc. Cambridge Philos. Soc. **39**, 159–167 (1943). [MF 9198]

The author transforms the equation of the quadric in S_n to a generalized focal form. Using this form, he writes the equation of a system of confocal quadrics in S_n and finds the numbers of quadrics of this system that satisfy, respectively, certain geometrical conditions. The tangential equation of this system is also obtained and discussed.
T. R. Hollcroft (Aurora, N. Y.).

Fano, Gino. Sulle forme cubiche dello spazio a cinque dimensioni contenenti rigate razionali del 4° ordine. Comment. Math. Helv. **15**, 71–80 (1943).

The present paper is concerned with those cubic primals of [5] which contain a rational normal ruled surface R^4 of order 4. If it has one such surface, it has ∞^2 similar ones [G. Fano, Comment. Math. Helv. **14**, 202–211 (1942); these Rev. **3**, 304]. Similarly, surfaces φ^5 of order 5 in [5] with elliptic sections (Del Pezzo surfaces) do not lie on the general cubic primal of [5]. If a given cubic primal contains one such surface, it contains ∞^5 such. These two properties are related. A cubic primal may contain an R^4 and also a φ^5. The two surfaces meet in 10 points [G. Fano, Atti Accad. Naz. Lincei. Rend. (6) **11**, 329–335 (1930)]. It is shown that the two poristic properties mentioned above persist. Various consequences of this fact are discussed.
V. Snyder (Ithaca, N. Y.).

Semple, J. G. Properties of certain cubic primals. Quart. J. Math., Oxford Ser. **15**, 26–33 (1944). [MF 10686]

Using both analytic and synthetic methods the author develops properties of the cubic primal V_7^3 in [8] defined by $|x_{ij}|=0$, $i, j=1, 2, 3$, and of its double locus D_4^6.
R. J. Walker (Aberdeen, Md.).

Gauthier, Luc. Sur certaines variétés cubiques rationnelles sans point double. C. R. Acad. Sci. Paris **216**, 435–437 (1943). [MF 10022]

In $[2n+1]$ the $2n$ dimensional cubic variety V_{2n}^3 which contains two $[n]$ without a common point is generally without a double point. It can be represented birationally upon a $[2n]$, the primal sections corresponding to quartic varieties. In the present paper it is shown more generally that the cubic V_{2n}^3 of $[2n+1]$ containing V_n^{n+2} or two $[n]$ of this Segre variety are rational and do not have multiple points. They are particular cases if $n>1$. When $n=2$, the varieties V_n^{n+2} and the V_n^{2n} coincide in quadric surfaces. The V_4^3 of [5] of Morin [Rend. Sem. Mat. Univ. Padova **11**, 108–112 (1940)] needs correction. In the notation there used, instead of $\lambda=0$, $\rho=0$, it should be $\lambda=1$, $\rho=9$. The procedure is poristic. In general the cubic variety of [4] does not contain a quartic surface. If, in particular, it contains one, it contains ∞^1. *V. Snyder* (Ithaca, N. Y.).

Apéry, Roger et Gauthier, Luc. Extension des transformations birationnelles des courbes de l'espace ordinaire à l'espace ambiant. C. R. Acad. Sci. Paris **217**, 129–131 (1943). [MF 10639]

Birational transformations between two plane curves cannot always be extended to their entire planes; besides the necessary invariants among the moduli of the curves, the genera of the successive adjoints must be respectively equal. The present note shows that, in the case of curves in spaces of higher dimensions, the equality of their genera and of their moduli are both necessary and sufficient to insure that they are birationally equivalent. A curve is expressed by its projecting cone and a monoid. The following theorem is established. Two birationally equivalent curves C, C' can be transformed into each other by Cremona transformations in such manner that a surface S passing simply through C may have contact of any given order with a given surface S' passing simply through C'. *V. Snyder.*

Du Val, Patrick. The Jacobian algorithm and the multiplicity sequence of an algebraic branch. Rev. Fac. Sci. Univ. Istanbul. Ser. A. **7**, 107–112 (1942). (English. Turkish summary) [MF 9553]

An algorithm is defined as follows. Given a set of positive integers a_i, let $d=a_j$ be the smallest and let q be the greatest integer such that $b_i=a_i-qa_j\geqq 0$, $i\neq j$. Then b_i, $b_j=a_j$, is the second stage of the algorithm, where all $b_i=0$ have been omitted. The algorithm is completed when the set is reduced to a single integer, the H.C.F. of the original set. In terms of this algorithm the author generalizes the classical decomposition of a branch of a plane algebraic curve, there being q successive d-fold points of the branch for each pair d, q. The first stage in this algorithm is the set of "characters" of the branch. A character is the sum of the multiplicities of the first n points of the branch, provided the nth point is proximate to no point to which its successor is not also proximate. Necessary and sufficient conditions are given that a set of integers be characters of a planar branch, but no such conditions are known for branches which do not lie in a plane. *R. J. Walker* (Aberdeen, Md.).

Châtelet, François. Sur la notion d'équivalence due à Poincaré. C. R. Acad. Sci. Paris **216**, 142–144 (1943). [MF 10006]

Let V and V' be algebraic varieties over an algebraically closed field F, defined by polynomials over a subfield P of F. Then V and V' shall be called Q-equivalent, where Q is a field between P and F, if there exists between them a birational correspondence, with coefficients in Q, such that no infinite set of points of V (or V') corresponds to a single point of V' (or V). The author states the following theorem: if V and V' are of dimension s and are F-equivalent, then they are Q-equivalent, where the degree of Q over P is a factor of $s+1$. *R. J. Walker* (Aberdeen, Md.).

Severi, Francesco. Sugl'integrali semplici di 1ª specie e sulle involuzioni irregolari appartenenti ad una varietà o superficie algebrica. Ann. Mat. Pura Appl. (4) **21**, 1–20 (1942). [MF 10504]

In a recent paper, F. Enriques [Univ. Nac. Tucumán. Revista **1**, 293–296 (1940); these Rev. **2**, 296] partly reaffirmed and partly contradicted the results obtained by F. Severi [Ann. Mat. Pura Appl. (1) **21**, 55–79 (1905)] on

Abel's theorem and its application to irregular algebraic surfaces. The present paper re-examines the earlier one and generalizes to algebraic varieties with the following results. On an algebraic variety V_k of surface irregularity $q>0$, every involution of irregularity q includes the fundamental involution of V_k or is compounded with it. If in this system there is an irreducible C_1 which has $q'<q$ integrals of the first kind constant along C_1, then either (a) C is isolated; (b) C belongs to a continuous system of analogous transitive systems which together comprise an isolated system along which the integrals of the first kind are constant; (c) C consists of an intransitive system composed of an infinity of transitive systems traced on the varieties of an involution of species greater than 1 and to a subsystem. On each of these varieties are q' integrals which remain constant. This procedure is applied to a surface of irregularity q. If it contains a continuous infinity of involutions of irregularity q, it also contains a pencil of genus p, with which these involutions are compounded, and the involutions themselves are referable to a ruled surface of genus p or to a pencil of genus p of elliptic curves. If F does not contain an infinity of involutions of the same irregularity q, it is birationally equivalent to a surface of Picard (simple or multiple), image of the first fundamental involution of $F(q=2)$. The involutions are birationally equivalent to a surface having an elliptic pencil of elliptic curves.

V. Snyder (Ithaca, N. Y.).

Finsler, Paul. Reelle Freigebilde. Comment. Math. Helv. 16, 73–80 (1944).

An algebraic variety G, lying in a projective space $[\rho]$ $(\rho\geqq 1)$, is intersected by every linear space $[r]$ of $[\rho]$ $(1\leqq r\leqq \rho)$ in a set of complex (or, in particular, real) points, which may be finite or infinite (in the second case, the set contains an algebraic curve or a higher variety). The author has previously considered and studied in the complex domain the case in which, for each space $[r]$ of $[\rho]$ intersecting the variety G in a finite set, this set consists of points which are linearly independent (and therefore no more than $r+1$ in number). [Cf. especially Comment. Math. Helv. 9, 172–187 (1937); 11, 62–76 (1938), where these varieties are called free-manifolds (Freigebilde), and, in particular, free-systems (Freisysteme) when they consist of a finite number of linear spaces.]

Here the free-manifolds are studied in the real domain. The author says that an algebraic variety (in particular, a free-manifold or a free-system) G is real when (i) G is self-conjugate (that is, contains the complex-conjugate of each of its points) and (ii) the complex linear space of minimum dimension containing all the real points of G also contains all the nonreal points of G. A real free-system consists of linear spaces, each of which is real. All the components of a 1-dimensional real free-manifold, as well as all their mutual intersections, are real. Every nonreal point of a real free-manifold G, of dimension not less than 1, lies on at least one real free-curve contained in G; the latter always contains an infinity of real points. If a real quadric Q of $[\rho]$, of equation $Q=0$, contains all the real points of a real free-manifold G of $[\rho]$, or, more generally, if Q contains $\rho+1$ linearly independent real points of G, and either $Q\geqq 0$ or $Q\leqq 0$ for all real points of G, then every (real or nonreal) point of G lies on Q. *B. Segre* (Manchester).

Differential Geometry

Humbert, Pierre. Géométrie plane dans l'espace attaché à l'opérateur Δ_3. J. Math. Pures Appl. (9) 21, 141–153 (1942). [MF 9301]

In his study of a certain partial differential equation of third order $\Delta_3 U=0$, Devisme [J. Math. Pures Appl. (9) 19, 359–393 (1940); these Rev. 3, 21] was led to the consideration of an associated three-dimensional space with linear-element defined by $ds^3=dx^3+dy^3+dz^3-3dxdydz$. The author studies the geometry of the xy-plane of this space. The motion group of this plane is the two parameter set generated by the symmetries in the lines of inclination 45°. The author defines and studies the geometry of the three isotropic directions, circles as the cubic curves through the three circular points at infinity, orthogonality, inversion and parabolas which are the ordinary cubic and semicubical parabolas. *J. DeCicco* (Chicago, Ill.).

Kasner, Edward and Kalish, Aida. The geometry of the circular horn triangle. Nat. Math. Mag. 18, 299–304 (1944). [MF 10690]

Kasner, Edward. Geometric properties of isothermal families. Publ. Inst. Mat. Univ. Nac. Litoral 5, 10 pp. (1943). [MF 10765]

Various geometric characterizations of plane families of isothermal curves are stated. The simplest of these is Lamé's theorem for an isothermal orthogonal net of curves. It states that an orthogonal net is isothermal if and only if the sum of the rates of variation of the curvatures of the two curves at each point with respect to their arc lengths is zero. A result similar to Lamé's theorem is obtained by considering a symmetric three-web, that is, three families of ∞^1 curves which cut each other at angles of $2\pi/3$. In this case, also, the symmetric three-web is isothermal if and only if the sum of the rates of variation of the curvatures with respect to arc length is zero. Another theorem states that the complete system of isogonal trajectories of a given one parameter family of curves F is linear if and only if F is isothermal. Still another result is that the angle between two independent isothermal families of curves is a harmonic function of x, y. Various other characterizations are given which are connected with the author's previously defined velocity systems. *A. Fialkow.*

Kasner, Edward and DeCicco, John. Scale curves in conformal maps. Proc. Nat. Acad. Sci. U. S. A. 30, 162–164 (1944). [MF 10794]

In connection with a conformal map of a surface Σ on a plane π, a scale curve on Σ or on π is the locus of points along which the scale function σ, the ratio of corresponding elements of arc length in Σ and in π, does not vary. In this paper the authors state, without proof, theorems concerning families of scale curves and apply the results to the cartography of the sphere. Of central importance are quasi-isothermal families, which are families of scale curves along each curve of which the Gaussian curvature of Σ is constant. Among the results are the following. If a quasi-isothermal family on π consists of parallel lines or concentric circles, then Σ is either developable or applicable to a surface of revolution. The only conformal map of a sphere on a plane with straight scale curves is the Mercator projection, and the only conformal maps of a sphere on a plane with circular scale curves are the Ptolemy stereographic and Lambert projections. *S. B. Myers* (Cambridge, Mass.).

DeCicco, John. Geometric properties of generalized dynamical trajectories. Publ. Inst. Mat. Univ. Nac. Litoral **5**, 7 pp. (1943). [MF 10764]

The paper deals with plane fields of force in which the force vector depends upon the lineal element (x, y, y'). It generalizes various theorems of Kasner concerning positional fields of force. There are ∞^3 trajectories associated with each such generalized field of force. It is shown that the locus of the foci of the osculating parabolas to the ∞^1 trajectories having a common lineal element is a circle passing through the common point. The angular rate λ is defined as the instantaneous rate of change of the inclination of the force vector with respect to the inclination of the corresponding lineal element. After defining "lines of force," it is shown that the ratio of the curvatures of any trajectory starting from rest at a point of the field and of the corresponding line of force is $(1-\lambda)/(3-\lambda)$. *A. Fialkow.*

Huron, Roger. Sur la torsion des courbes gauches. C. R. Acad. Sci. Paris **216**, 791–792 (1943). [MF 10653]

The author states various theorems concerning (1) a space curve and its osculating circular helices, (2) the contact between a space curve and a surface. *A. Fialkow.*

Gürsan, Feyyaz. L'élément infinitésimal d'ordre supérieur d'une courbe gauche. Rev. Fac. Sci. Univ. Istanbul (A) **6**, 27–35 (1941). (French. Turkish summary) [MF 10812]

Cesàro showed that the differential element of order n of a plane curve may be given by a polygonal line whose vertices are the point of the curve and the centers of curvature of the consecutive evolutes. Later von Mises [C. R. Acad. Sci. Paris **206**, 1338–1340 (1938)] pointed out that replacing the centers of curvatures by polar lines and the evolutes by the edges of regression of the polar surfaces leads to a configuration which is not sufficient to determine higher differential elements, and indicated a way out of the difficulty. The author of the present paper gives a more elegant solution of the problem by using together with the given curve its adjoint, that is, a curve whose curvature and torsion are equal, respectively, to the torsion and curvature of the given curve. The general theory is illustrated on the examples of helices and Bertrand curves. Formulas (20) contain easily corrected misprints.

G. Y. Rainich (Ann Arbor, Mich.).

Müller, Hans Robert. Über die Striktionslinien von Kurvenscharen. Monatsh. Math. Phys. **50**, 101–110 (1941). [MF 10474]

A point P on a curve C of a family of curves is said to belong to the "line of striction" of the family if it furnishes an extremum to the distance from points of C to neighboring curves of the family. By simple calculations, theorems are proved concerning such lines. For example, a line of striction of a family F of geodesics is characterized by the fact that the tangents to the curves of F at the points of C are parallel with respect to C in the sense of Levi-Civita. *S. B. Myers* (Cambridge, Mass.).

Gerretsen, J. C. H. Die Liniengeometrie des 4-dimensionalen Raumes. I. Nederl. Akad. Wetensch., Proc. **45**, 690–696 (1942). [MF 10434]

This is the first paper of a series in which the author proposes to develop line geometry in S_4 by use of the Grassmann representation. The author is concerned here with the linear spaces on the Grassmannian variety and the representation of systems of lines of S_4 satisfying various incidence conditions. The results in this introductory paper do not appear to the reviewer to be novel, although they are presented in a systematic and elegant fashion.

J. A. Todd (Cambridge, England).

Finikoff, S. Sur le problème de S. Bachvaloff dans la théorie des couples stratifiables. Rec. Math. [Mat. Sbornik] N.S. **12(54)**, 287–314 (1943). (French. Russian summary) [MF 10224]

Two congruences (r_1), (r_2) generated by a pair of lines r_1, r_2 are said to be "stratifiable" if there exist surfaces (P_1), (P_2), generated by points P_1, P_2 on r_1, r_2, respectively, such that the tangent plane to S_i at P_i passes through r_j $(i\neq j)$. The object of this paper is further to elucidate and extend the results of S. Bachvaloff on such pairs of congruences [Rec. Math. [Mat. Sbornik] (N.S.) **6(48)**, 67–76 (1939); C. R. (Doklady) Acad. Sci. URSS (N.S.) **23**, 743–745 (1939); these Rev. **1**, 270]. Let (r) be the congruence generated by the common normal r to r_1, r_2. Attached to each line r is a local trihedral of reference, the x_3-axis being the line r, the origin the midpoint of the segment formed by the focal points on r, the x_1x_3- and x_2x_3-planes the bisectors of the focal planes of r. Let α_i be the angles between r_i and the x_1-axis, and h_i the distance of the intersection of r_i with r from the origin. The quantities α and h defined by $2\alpha=\alpha_1+\alpha_2$, $2h=h_1+h_2$ are called, respectively, the asymmetry and the eccentricity of the pair. It is shown that a congruence (r) which has associated with it a stratifiable pair of congruences depends ordinarily on an arbitrary function of two independent variables. It is shown, too, that, if the asymmetry is zero, then either the eccentricity is zero or the rays r_1, r_2 are orthogonal. A pair of stratifiable congruences is said to be symmetric if the asymmetry and the eccentricity are zero. In this case the congruence (r) depends on six arbitrary functions of one variable. To each such congruence (r) there is associated a single infinitude of symmetric pairs of congruences. Several special cases are considered. *V. G. Grove* (East Lansing, Mich.).

***Blaschke, Wilhelm und Bol, Gerrit. Geometrie der Gewebe. Topologische Fragen der Differentialgeometrie.** J. W. Edwards, Ann Arbor, Michigan, 1944. viii+339 pp. $9.25.

The original appeared as vol. 49 of Die Grundlehren der Mathematischen Wissenschaften in Einzeldarstellungen mit Besonderer Berücksichtigung der Anwendungsgebiete; publisher: J. Springer, Berlin, 1938.

Bol, G. Doppelverhältnisse im Fünfgewebe. Ann. Mat. Pura Appl. (4) **21**, 21–24 (1942). [MF 10505]

The following is an adaptation of the introduction. Wenn die Kurven eines Viergewebes Tangenten besitzen, so ist das Doppelverhältnis der vier Tangenten an die Gewebekurven durch einen Punkt invariant bei allen stetig differenzierbaren topologischen Abbildungen. Ist diese Invariante eine Ortsfunktion, so bilden die Kurven, längs denen das Doppelverhältnis fest bleibt, (im Kleinen) eine fünfte Kurvenschar, die "Doppelverhältnisschar" des Viergewebes. In einem Fünfgewebe sei jede Schar Doppelverhältnisschar der vier andern. Dann lässt sich das Gewebe auf vier Geradenbüschel und die Kegelschnitte durch ihre vier Scheitel topologisch abbilden, also auf ein "Ausnahme-Fünfgewebe" [Blaschke und Bol, Geometrie der Gewebe, J. Springer, Berlin, 1938, pp. 108, 110]. "In dieser Note wird gezeigt, dass von den vorausgesetzten fünf Bedingungen

nur drei unabhängig sind. Es gilt nämlich: sind in einem Fünfgewebe drei der Scharen Doppelverhältnisscharen, so lässt es sich auf ein Ausnahme-Fünfgewebe abbilden. Durch Beispiele wird gezeigt, dass zwei der drei Bedingungen nicht ausreichen. Ein Sonderfall dieses Satzes wurde von Pantazi bewiesen [C. R. Acad. Sci. Roum. **1937**, 278].

P. Scherk (Saskatoon, Sask.).

Graf, H. und Thomas, H. Zur Frage des Gleichgewichts von Vierecksnetzen aus verknoteten und gespannten Fäden. I. Math. Z. **48**, 193–211 (1942). [MF 8547]

Consider an ideal physical model of a portion of an analytic surface, and on it a net consisting of two finite systems of ideal strings tied together by knots. Along the boundary of the surface we imagine forces acting on the strings so that the strings remain in equilibrium and on the surface. In general there will result a normal pressure between strings and surface. The authors investigate when and how it is possible to pass to the limit of two continuous families of curves in such a way that the normal force tends to zero. In the limit the two systems of curves will cover the surface and retain their shape even if the latter be removed. Special cases are considered. *W. Feller.*

Wilcox, L. R. An application of a theorem of Sylvester. Amer. Math. Monthly **51**, 270–273 (1944). [MF 10565]

The theorem that the tangent plane to a developable surface is the same at all points of a fixed generator is extended to n-dimensional space in the following form. In a projective space S_n let an increasing sequence of subspaces

$$S_1 \subset S_2 \subset \cdots \subset S_{n-1} \subset S_n$$

be given. Let α_{hk} be the operation of passing from a curve C in S_k (but not in S_{k-1}) to a curve in S_h ($h<k$) by intersecting S_h with the $(k-h)$-dimensional osculants of C. If C is a curve in S_n for which each $\alpha_{hn}C$ exists and is not in S_{h-1}, then

$$\alpha_{hh_1}\alpha_{h_1h_2}\cdots\alpha_{h_rn}C=\alpha_{hn}C,$$

where $h<h_1<\cdots<h_r<n$. The proof is furnished by means of a complicated Sylvester theorem on determinants.

A. J. Kempner (Boulder, Colo.).

Chang, Su-Cheng. On the surfaces of coincidence. Bull. Amer. Math. Soc. **49**, 900–903 (1943). [MF 9687]

Given a nonruled analytic surface M. It is proved that there are six and only six asymptotic osculating quadrics Q_j associated with a point P of M such that each Q_j contains the four consecutive asymptotic tangents of one system along a curve T of M through P. Using the fact that T is tangent to a direction of Darboux at P, it is then proved that when and only when M is a surface of coincidence each Q_j is an osculating quadric of an asymptotic ruled surface along a curve of Darboux. *T. R. Hollcroft.*

Bieri, Hans. Invariante Herleitung der Differentialgleichungen für 3-fache Orthogonalsysteme. Comment. Math. Helv. **15**, 287–295 (1943).

The surfaces of a triply orthogonal system have the well-known property of intersecting in lines of curvature. The unit tangent vectors of these curves of intersection form a triple of mutually orthogonal vectors. The present paper is chiefly devoted to a derivation of explicit formulas for the derivatives of these vectors with respect to arc length along the intersections (lines of curvature). The coefficients which appear are geodesic curvatures of the intersections and principal curvatures of the various surfaces along the intersections. *G. A. Hedlund.*

Bernstein, S. Complément à mon article "Renforcement du théorème des surfaces à courbure négative." Bull. Acad. Sci. URSS. Sér. Math. [Izvestia Akad. Nauk SSSR] **7**, 297–298 (1943). (Russian) [MF 10943]

The paper appeared in the same Bull. **6**, 285–290 (1942); these Rev. **5**, 14.

Scherrer, W. Eine Formel für die geodätische Krümmung. Comment. Math. Helv. **16**, 101–104 (1944).

The well-known Gauss-Bonnet formula

$$\int_\gamma \kappa_g ds = 2\pi - \iint_\sigma K d\omega,$$

where κ_g is the geodesic curvature of a simple closed curve γ on a simply connected surface, K is the Gaussian curvature and σ is the interior of γ, is usually derived intrinsically. The present paper gives a neat derivation of this result for the case of a surface in E_3 by a nonintrinsic method in which the spherical image of the surface plays an important part. As a by-product, the theorem of Jacobi to the effect that under suitable restrictions the principal normal image of a closed curve in E_3 divides the unit sphere into two parts of equal areas is attained. *G. A. Hedlund.*

Preissman, Alexandre. Quelques propriétés globales des espaces de Riemann. Comment. Math. Helv. **15**, 175–216 (1943).

This paper contains new results and a report on known facts on the general problem of relations between the geometric and topological properties of a Riemannian manifold (R.m.); the main topic is the behavior of the geodesics. Chapter I discusses normal coordinates, Fermi's coordinates along a geodesic and, in much detail, expressions for the first and second variation of the arc length following Synge's treatment. Chapter II discusses the notions of complete R.m. (following Hopf and Rinow), covering manifolds, the notion of endpoint (as defined by Freudenthal), the notion of pole (a point P is a pole if every geodesic ray issuing from P is a minimal geodesic, that is, furnishes an absolute minimum of the distance from P to any other point on it).

Chapter III studies R.m.'s of negative curvature. In the simply connected case every point is a pole. With a point P as vertex and a geodesic g not through P as directrix one can then construct the geodesic cone. By studying the geometry on that cone it is shown that g has exactly one point of minimum distance from P and that the two halves of g diverge to infinity monotonically. Another result is that in a geodesic triangle the sum of the angles is less than π; this was known only for small triangles. It follows that an isometry of space can transform at most one geodesic in itself and that every other isometry commuting with it leaves the same geodesic invariant. By considering covering manifolds one derives that every Abelian subgroup of the fundamental group of a compact R.m. is cyclic. Topological products of two compact manifolds cannot therefore be metrized with negative curvature. On the other hand, the fundamental group of a compact R.m. is shown not to be cyclic, and therefore not even Abelian; this was known in the case of constant curvature. Another result: any (free) homotopy class contains at most one closed geodesic. Chapter IV studies R.m.'s of positive curvature. Proofs are

given for (a) the generalized Bonnet theorem on a space with a possible l.b. for the curvature, (b) Synge's theorem that an orientable space of even dimension is simply connected and related theorems, (c) Cohn-Vossen's theorem that there exists only one endpoint. In chapter V, Mangoldt's theorem for surfaces that the poles form a bounded set is carried over to any dimension. Finally, it is shown that if there is a pole P then every geodesic not through P behaves with respect to P as in the case of negative curvature [see above]; in particular, there are no closed geodesics.

H. Samelson (Syracuse, N. Y.).

Garnier, René. Sur l'existence de relations entre des fonctions contiguës de Painlevé. C. R. Acad. Sci. Paris **217**, 60–62 (1943). [MF 10634]

The solution of the problem of Plateau assures the existence, for an arbitrary simple closed space curve C, of a minimal surface S of the type of the circular disc, bounded by C. The solution is not always unique. It has been stated without proof by H. A. Schwarz and repeated by G. Darboux that if C is a skew quadrilateral then C cannot bound more than one minimal surface of the type of the circular disc and having no singularities in its interior. The above uniqueness theorem follows from the more general result established by T. Radó [Proc. Nat. Acad. Sci. U. S. A. **16**, 242–248 (1930)]: if a simple closed space curve C has a simply covered convex curve as its parallel or central projection upon some plane, then C cannot bound more than one minimal surface S of the type of the circular disc; such a surface S cannot have any branch points or even intersect itself in the large.

The present paper gives an alternative proof, by means of the theory of differential equations, of the above result concerning skew quadrilaterals. *E. F. Beckenbach.*

Monna, A. F. Sur quelques propriétés d'une classe de surfaces minima. Nederl. Akad. Wetensch., Proc. **45**, 681–686 (1942). [MF 10432]

Let $x_j(u, v)$, $j=1, 2, 3$, be a triple of conjugate harmonic functions in the finite simply-connected domain D; that is, the x_j are harmonic and satisfy the differential equations $E=G$, $F=0$, so that the x_j map D conformally on a minimal surface. Then $x_j(u, v)=\Re f_j(z)$, $z=u+iv$, where $f_j(z)$ is analytic in D; and $\sum[f_j'(z)]^2=0$. Many known properties of univalent functions $f(z)$ do not hold for analytic functions generally. Accordingly, since the plane is a minimal surface, analogously formulated properties cannot hold for conformal maps on all minimal surfaces, but only on a suitably defined class of minimal surfaces.

If the above functions $f_j(z)$ are univalent, $j=1, 2, 3$, then the corresponding minimal surface is said to be univalent [cf. Beckenbach, Amer. J. Math. **54**, 718–728 (1932)]. The author shows that the results concerning univalent functions $f(z)$ in two recent papers of J. Wolff [Nederl. Akad. Wetensch., Proc. **44**, 956–962 (1941); C. R. Acad. Sci. Paris **213**, 158–160 (1941); these Rev. **5**, 36] hold also for univalent minimal surfaces. A typical result is the following. Let Γ_α denote a rectifiable Jordan arc lying in $|z|<1$ except for its endpoint α on $|z|=1$ and having the property that all its chords make angles with the radius joining the origin to α of magnitude less than $\pi/2-\epsilon$ $(0<\epsilon<\pi/2$, ϵ otherwise arbitrary but fixed). If $x_j(u, v)$, $j=1, 2, 3$, are a triple of bounded univalent conjugate harmonic functions for $|z|<1$, then, for almost all points α of $|z|=1$, the image of Γ_α on the minimal surface is rectifiable.

E. F. Beckenbach (Austin, Tex.).

Pyle, H. Randolph. Conformal mapping of surfaces. Duke Math. J. **11**, 369–371 (1944). [MF 10675]

If $x'=\phi(x, y)$, $y'=\psi(x, y)$ is a mapping M of a surface S with coordinates (x, y) on a surface S' with coordinates (x', y'), the classical conditions for conformality of M are quadratic in the partial derivatives of ϕ and ψ. In this paper the author reduces these quadratic conditions to linear ones.

S. B. Myers (Cambridge, Mass.).

Haantjes, J. Conformal differential geometry. II. Curves in conformal two-dimensional spaces. Nederl. Akad. Wetensch., Proc. **45**, 249–255 (1942). [MF 10392]

In a former paper [same Proc. **44**, 814–824 (1941); these Rev. **3**, 189] the author made a study of curves in Euclidean n-space for $n>2$. His approach was through the invariant theory of the change of Euclidean metric $a_{ij}'=\sigma^2 a_{ij}$. For the case $n=2$ it is found desirable to strengthen the condition on σ which insures flatness so as to insure invariance of circularity. When $n>2$ the former implies the latter. With this amendation the author's previous development $n>2$ can be specialized immediately to $n=2$. There results a conformal parameter of the third order (τ), a conformal differentiation along a curve (involving the Christoffel symbols and a quantity Q_i defined along the curve) and conformal Frenet-Serret formulas. In the latter a fifth order conformal invariant $h(\tau)$ appears, the inversion curvature, which determines the curve to within transformations of the Möbius group. Curves of constant h are loxodromes. Several geometric interpretations of τ, Q_i and h are developed, of which we exhibit two. It is found that the quantities Q_i define uniquely at each point of the given curve a coaxial system of circles passing through that point, one of which is the osculating circle of the curve. There is just one loxodrome through a point P of a curve and having six point contact. It meets a certain pencil of circles under a constant angle α. Then $h(P)=\cot 2\alpha$.

J. L. Vanderslice (College Park, Md.).

Haantjes, J. Conformal differential geometry. III. Surfaces in three-dimensional space. Nederl. Akad. Wetensch., Proc. **45**, 836–841 (1942). [MF 10439]

[Cf. the preceding review.] As in the previous two papers, the attack is through the invariant theory of the change of Euclidean metric (A) $g_{ij}'=\sigma^2 g_{ij}$. The use of pentaspherical coordinates is avoided. If $l=\frac{1}{2}(k_1+k_2)$, $\rho=\frac{1}{2}|k_1-k_2|$, where k_i are the principal curvatures of a surface S_2 in R_3, and if g_{ij}, $a_{\alpha\beta}$, $h_{\alpha\beta}$ are the coefficients of the metric in R and the first and second fundamental forms of S, then $G_{ij}=\rho^2 g_{ij}$, $A_{\alpha\beta}=\rho^2 a_{\alpha\beta}$, $H_{\alpha\beta}=\rho(h_{\alpha\beta}-la_{\alpha\beta})$ are invariant tensors under (A) and form the basis of the conformal surface theory; $A_{\alpha\beta}$ is used to define conformal metric and covariant differentiation on S. A covariant differentiation is also found in R but its parameters are defined only at points of S. However, it induces on S via the normal the covariant differentiation defined by $A_{\alpha\beta}$. Identities among the fundamental surface quantities and some geometrical interpretations are given. For example, the null directions of $H_{\alpha\beta}$ bisect the principal directions while the elementary conformal arc length defined by $A_{\alpha\beta}$ is the angle between the "central" spheres at neighboring points of S.

J. L. Vanderslice (College Park, Md.).

Haantjes, J. Conformal differential geometry. IV. Surfaces in three-dimensional space. Nederl. Akad. Wetensch., Proc. **45**, 918–923 (1942). [MF 10442]

[Cf. the two preceding reviews.] Since the parameters of the covariant differentiation on R_3 are known only on S_2,

not all the components of the corresponding curvature tensor are well defined. However, those that are can be broken down into combinations of quantities of lesser index order. One of these is the third fundamental conformal tensor $V_{\alpha\beta}$, another a quantity V_α depending on the surface normal. The curvature tensor for the intrinsic covariant differentiation on S introduces in a familiar way a curvature scalar K. Various sets of relationships among these fundamental quantities are derived, among them the analogues of the Gauss-Codazzi equations. This leads to a proof of the fundamental theorem of conformal surface theory: when $A_{\alpha\beta}$, $H_{\alpha\beta}$, $V_{\alpha\beta}$ satisfy the above relationships there exists a surface unique to within conformal representations for which these tensors give the three fundamental forms. Provided a certain scalar does not vanish, $V_{\alpha\beta}$ is determined by $A_{\alpha\beta}$ and $H_{\alpha\beta}$. It is shown how a unique normal circle may be defined at each point of S, leading to a geometrical definition of the conformal geodesics previously defined as extremals of the conformal metric. They are the curves whose osculating circle at every point lies together with the normal circle on one sphere. The paper ends with a characterization of a surface whose normal circles pass through a fixed point. Throughout this and the preceding papers certain parallelisms with the work of Blaschke and others are duly noted by the author. *J. L. Vanderslice.*

Hsiung, Chuan-Chih. Projective invariants of intersection of certain pairs of surfaces. Bull. Amer. Math. Soc. **50**, 437–441 (1944). [MF 10613]

Let S_1, S_2 be two surfaces through a point O with distinct tangent planes t_1, t_2 intersecting in a line t. In a previous paper [Bull. Amer. Math. Soc. **49**, 877–880 (1943); these Rev. **5**, 158] the case in which t coincided with an asymptotic tangent of S_1 or S_2 was excluded. The present paper takes up this case. If t coincides with an asymptotic tangent of S_1, but not S_2, the equations of S_1 and S_2 may be written in the form $y=lxz+px^3+rx^2z+sxz^2+qz^3+\cdots$, $z=mx^2+ny^2+\cdots$. It is shown that the functions $I=lm/p$, $J=p^2q^4m^3/(l^{12}n^3)$ are projective invariants. In the case in which t is also an asymptotic tangent of S_2, the equation of S_2 may be written as $z=mxy+\cdots$. In this case $I=p^4q^3/(l^9m^4)$ is a projective invariant. The invariants in both cases are characterized geometrically in terms of cross-ratios. In each case the elements arising in the characterization arise from neighborhoods of the second order of one surface, and the third order of the other.

V. G. Grove (East Lansing, Mich.).

Spencer, Domina Eberle. Geometric figures in affine space. J. Math. Phys. Mass. Inst. Tech. **23**, 1–23 (1944). [MF 10115]

The purpose of this paper is to determine a classification of Study's dynamen from the point of view of tensor analysis. According to the author, the solution to this problem lies in the fact that "although Study worked in metric space, his figures are inherently affine." Hence, in order to determine the "affine ancestors of all the figures of Study," the author is led to an investigation of the geometric figures of affine space. The tensor interpretation of Study's figures in metric space is to be discussed in another paper.

First, the author discusses some of the fundamental concepts of three dimensional projective space by use of homogeneous coordinates. Points are introduced by means of contravariant pseudovectors, planes by means of covariant pseudovectors, lines by means of covariant and contravariant bivectors. In order to introduce an affine space, the the author singles out a plane, called the improper plane; its intersection with a plane or a line is denoted as improper line or point, respectively. Thus, parallel lines or planes may be defined. To the improper plane is assigned the absolute covariant vector $a_\lambda(1, 0, 0, 0)$ and hence to every contravariant vector x^λ may be assigned a weight number $x^0=x^\lambda a_\lambda$. With the aid of this weight concept, the author discusses the fundamental figure represented by a contravariant vector, that is, the weighed point.

In the remainder of the paper, the general procedure is as follows. Either alternating products of vectors of the same kind or direct products of vectors of different types are formed. Then the author studies the figures representing these combinations. For the alternating product of contravariant vectors, these figures are classified according to whether the Plücker relation is satisfied or not. A further subclassification is given by use of the terms proper, semiproper, improper. The result of the addition of various types of figures as well as the laws satisfied by addition is discussed. A similar procedure, with proper modification for the types of subclasses, is followed for the alternating product of two covariant vectors. Correspondences between these covariant and contravariant figures are discussed by introducing an E tensor. Next, the author studies the types of figures associated with the direct product of covariant and contravariant vectors. This leads to six distinct tensors of rank one. Furthermore, the sums of such figures lead to tensors of rank four. The resulting figures are classified in considerable detail. Finally, the author analyzes the various figures of Study and shows that the figures studied in this paper are essentially their affine ancestors. In an appendix, a nomenclature is offered for the figures which were studied in the paper. *N. Coburn.*

Spencer, Domina Eberle. The tensor representation of the figures of Study's "Geometrie der Dynamen." J. Math. Phys. Mass. Inst. Tech. **23**, 103–115 (1944). [MF 10630]

This is a continuation of the paper reviewed above. A metric tensor of rank three is introduced into the three dimensional Euclidean affine space. With the aid of this metric tensor the author's figures can be classified not only with respect to support and polarization (concepts introduced in the preceding paper) but also with respect to metric normalization. First, Study's "Stab" (figure consisting of two proper points) is discussed. By assigning a proper tensor representation and metric normalization to the author's figure of two proper points, it is shown that this figure coincides with Study's "Stab." The other Study figures are discussed in a similar manner. [The reviewer does not understand why the metric normalization is denoted as a tensor.]

It is noted that the use of the Study figures involves some complications in their tensor representations. It is the author's belief that support, polarization and metric normalization are the significant concepts in classifying Study's figures. Hence, the author assigns direct metric normalizations to various figures studied in the earlier paper. The resulting figures are considered as offering the simplest tensor representations of the Study figures.

N. Coburn (Austin, Tex.).

Bruck, Richard H. and Wade, T. L. The number of independent components of the tensors of given symmetry type. Bull. Amer. Math. Soc. 49, 470–472 (1943). [MF 8402]

The author's problem is to determine the number of components of each symmetry type into which an arbitrary covariant tensor $T_{i_1\cdots i_p}$ (in n dimensions) can be decomposed. Thus, if

$$(1)\, T_{i_1\cdots i_p[p]} = T_{i_1\cdots i_p} + \cdots + {}_{[\alpha]}T_{i_1\cdots i_p} + \cdots + {}_{[1^p]}T_{i_1\cdots i_p},$$

and if C_α denotes the number of components of ${}_{[\alpha]}T_{i_1\cdots i_p}$, then

$$(2) \qquad n^p = C_{[p]} + \cdots + C_{[\alpha]} + \cdots C_{[1^p]}.$$

For $p=2, 3, 4$, J. A. Schouten [Der Ricci-Kalkul, Springer, Berlin, 1924, chap. 7] has obtained expressions for the C_α. The author's method is to find these C_α in terms of n by use of the character table for the symmetry group on p letters. This is accomplished by showing that $C_\alpha = \Omega_\alpha$, where Ω_α is the rank of the immanent tensor $I^{(\alpha)}_{(j)}$ and using a known formula for Ω_α [Amer. J. Math. 64, 734–752 (1942); these Rev. 4, 128]. Specific computation of the C_α is given for $p=3, 4, 5, 6$. *N. Coburn* (Austin, Tex.).

Schenberg, Mario. On Cartan's integral invariants. Anais Acad. Brasil. Ci. 16, 9–12 (1944). (Portuguese) [MF 10888]

Cartan's formulation of the principle of motions [cf. E. Cartan, Leçons sur les invariants intégraux, Gauthier-Villars, Paris, 1922] is given in a slightly different form. *S. Chern* (Princeton, N. J.).

Seetharaman, V. On the existence of a metric for higher path-spaces. Proc. Indian Acad. Sci., Sect. A. 19, 167–176 (1944). [MF 11014]

A higher path-space is a space with coordinates $(x^1, \cdots, x^n)$ in which there is given a system of differential equations

$$(S) \quad d^{(\sigma+1)}x^i/dt^{(\sigma+1)} + \Gamma^i(t, x, x', \cdots, x^{(\sigma)}) = 0, \qquad i=1, \cdots, n;\ \sigma \geqq 2,$$

whose integral curves are called the paths. The author establishes a necessary and sufficient condition that the paths of (S) are the extremals of a regular problem of the calculus of variations, that is, they can also be defined by

$$\delta \int f(t, x, \cdots, x^{(\sigma)})dt = 0,$$

for a certain function f. The condition is that the equations of variation of (S) be self-adjoint. Proof is by induction. *S. Chern* (Princeton, N. J.).

MECHANICS

Hadwiger, H. Über Massenpunktverteilungen konstanter Trägheit auf der Kugel. Z. Angew. Math. Mech. 23, 61–62 (1943). [MF 9786]

The author determines a discontinuous mass-distribution on the unit sphere with the property that its moment of inertia is the same for any axis passing through the center of the sphere. *W. Feller* (Providence, R. I.).

Kincaid, W. M. and Morkovin, V. An application of orthogonal moments to problems in statically indeterminate structures. Quart. Appl. Math. 1, 334–340 (1944). [MF 9910]

In structures which are statically indeterminate of degree N, various methods of approximation such as "moment balancing" or systematic relaxation of joints have been employed. For each new loading system the calculations are begun afresh. The present writers construct basic orthogonalized moment systems dependent only upon the framework itself, from which the required quantities are readily obtained for any type of loading for that particular framework. *D. L. Holl* (Ames, Iowa).

Neville, E. H. Exercises on a tightrope. Philos. Mag. (7) 35, 414–419 (1944). [MF 11003]

The problem discussed is the shape of a uniform cord hanging in festoons through several smooth rings, when the cord is nearly tight so that each festoon is not far from its chord. Parameters are set up in terms of which power series developments can be found and the details carried far enough to give a simple formula for a first approximation to the separate sags. *P. Franklin* (Cambridge, Mass.).

Benedikt, Elliot T. On the representation of rigid rotations. J. Appl. Phys. 15, 613–615 (1944).

Consider a rigid body free to rotate about a point O. Let P_0 and P_1 be the positions occupied by a generic point, respectively, before and after a rotation through an angle θ about an axis parallel to the unit vector $\bar{h}$. It is then shown that

$$\overrightarrow{OP_1} = (e^{\theta h \times})\overrightarrow{OP_0},$$

where $\times$ is the usual symbol for vector multiplication and the expression in the parentheses represents the linear operator to be obtained in the indicated way from the formal expansion of the exponential function. Since the desired rotation can be generated in θ units of time by revolving the body about the given axis with unit angular velocity, it is clear that $\overrightarrow{OP_1}$ is that solution of the differential system

$$d\overrightarrow{OP_1}/d\theta = h \times \overrightarrow{OP_1}$$

which, for $\theta=0$, reduces to $\overrightarrow{OP_0}$. The stated result follows immediately. The author's proof, however, avoids the differential system by manipulating the elementary formulas (attributed to Euler) which give the components of $\overrightarrow{OP_1}$ in terms of the components of $\overrightarrow{OP_0}$. *D. C. Lewis.*

Bloch, A. A new approach to the dynamics of systems with gyroscopic coupling terms. Philos. Mag. (7) 35, 315–334 (1944). [MF 10886]

The author treats the elementary theory of linear dynamical systems, with grysocopic couplings between some of their degrees of freedom, by means of concepts and methods derived from the theory of electrical circuits. Much of the discussion is concerned with a transformation which carries a given dynamical system into a certain "reciprocal" system; considerable attention is given to electrical analogues of the usual kind. Several examples are discussed in some detail. Actually, the approach employed by the author has little, if any, novelty. *L. A. MacColl.*

Agostinelli, Cataldo. Sulla trasformazione delle equazioni della dinamica. Ann. Mat. Pura Appl. (4) 21, 39–109 (1942). [MF 10507]

This paper deals with the following problem. Given two dynamical systems, each having n degrees of freedom, and each having connections which are independent of the time, under what conditions do the systems have ∞^{2n-k-1} trajectories in common, or, if there are no applied forces, under what conditions do the systems have ∞^{2n-k-2} trajectories in common? The case in which $k=0$ and there are no applied forces was solved by Levi-Civita. The case in which $k=0$ and there are applied forces has been solved previously by Agostinelli [Ist. Lombardo, Rend. (3) 70, 253–260 (1937)]. In the present paper the author first discusses the general problem at considerable length, and then he gives a detailed solution of the case in which $k=1$. The results are complicated, and they do not admit of a concise summary. *L. A. MacColl* (New York, N. Y.).

Keller, Ernest G. Some present non-linear problems of the electrical and aeronautical industries. Quart. Appl. Math. 2, 72–86 (1944). [MF 10338]

The author gives a number of examples from industry of nonlinear problems involving systems with a finite number of degrees of freedom. The examples are selected so as to illustrate the several distinct methods available to solve such problems. In most cases a fundamental part of the problem is the determination of the method which will yield a reasonable answer in a reasonable amount of time. Perturbation methods and successive approximations are exploited.

The author considers specifically nonlinear control circuits, nonlinear transmission line phenomena, nonlinear springs, electric locomotive oscillations, dynamic braking of a synchronous machine, a double-valued nonlinear problem and a nonlinear problem of two oleo-pneumatically coupled masses one of which is subject to impact. In the case of the electric traction problem the author stresses the necessity of checking results arrived at by improvised methods (and there are really no others in the case of many of these problems) with experimental tests. *N. Levinson.*

✱Bouffard, Jean. Balistique extérieure. Nouvelle méthode de calcul et d'étude de la trajectoire d'un projectile. Actual. Sci. Ind., no. 907. Hermann et Cie., Paris, 1942. 78 pp.

✱Bliss, Gilbert Ames. Mathematics for Exterior Ballistics. John Wiley and Sons, Inc., New York, 1944. vii+128 pp. $2.00.

This book treats exterior ballistics strictly as a problem in particle dynamics, that is, rotational effects are ignored and resistance is assumed to act along the tangent to the trajectory. The field of ballistics, thus restricted, is presented with exceptional clearness and thoroughness and a surprising amount of information is packed into a little book of only 128 pages.

The introductory chapter presents the problem of ballistics from the point of view of the battery commander in the field, with considerable attention to military maps, firing data and the use of range tables. In chapter II the differential equations are set up for standard conditions, the drag function and its experimental background are explained and the standard air density function is discussed. Chapter III covers the essentials of the Siacci theory. Chapter IV develops Moulton's method of numerical integration set up especially for computing trajectories and also gives an account of the differential analyzer and its use in ballistics. Chapter V on differential corrections is the longest chapter in the book and includes the author's own contributions to the field of ballistics. It is interesting to observe how the subject of adjoint differential equations, once an esoteric theory of "pure" mathematicians, has now become a commonplace tool of practical computers. The last chapter gives an interesting and mainly descriptive account of the problem of bombing from airplanes. A feature of the book enhancing its practical usefulness is a set of tables giving the drag function, its derivative and the standard air density function. A typical trajectory together with differential corrections is also included.

After searching for something to criticize the reviewer can only observe that in respect to their effects on the trajectory the rotation and oscillations of the projectile, subjects omitted entirely, are considerably more important than the sphericity and rotation of the earth, which were treated in some detail. *W. E. Milne* (Corvallis, Ore.).

Hydrodynamics, Aerodynamics

Dolidze, D. E. On the general linear problem in hydrodynamics. Bull. Acad. Sci. Georgian SSR [Soobščenia Akad. Nauk Gruzinskoi SSR] 3, 649–656 (1942). (Russian. Georgian summary) [MF 10329]

The boundary value problem for the three-dimensional nonstationary motion of a viscous incompressible fluid in the interior or exterior region bounded by a regular closed surface, with preassigned initial and boundary values of the velocity, is reduced to an integral equation problem of Volterra type. Existence and uniqueness theorems are established. A slight defect in the author's previously given solution of the corresponding two-dimensional problem [Mitt. Akad. Wiss. Georgischen SSR [Soobščenia Akad. Nauk Gruzinskoi SSR] 2, 43–49 (1941); cf. these Rev. 3, 219] is pointed out; the present solution is free from this defect. *E. F. Beckenbach* (Austin, Tex.).

Nath Sharma, Pirthvi. Elliptic sources and vortices. Proc. Benares Math. Soc. (N.S.) 4, 33–38 (1943). [MF 10346]

It is known that, in general, a desired incompressible flow can be obtained by a suitable distribution of point sources, sinks and vortices. However, even for the flow past the simple case of an elliptic boundary, an infinite number of sources and sinks are required. Much of the work in a specific problem, such as the flow past an elliptic body that the author treats, can be simplified if special types of sources, sinks and vortices are introduced. An elliptic source (or sink) is defined as a line segment source whose velocity potential ϕ satisfies the following conditions. (a) The flow across an elliptic boundary having the end points of the source as foci is constant. (b) The velocity due to it is constant over any confocal elliptic boundary. (c) The velocity due to it at infinity is zero. An elliptic doublet is obtained when an elliptic source and an elliptic sink have their midpoints indefinitely close. Following Kirchhoff, the elliptic vortex is defined. Elliptic coordinates are used to advantage. Flows with circulation are treated and images are studied. *A. Gelbart* (Syracuse, N. Y.).

Banerji, A. C. and Varma, R. S. On the motion of a compressible ellipsoid in a viscous fluid. Proc. Benares Math. Soc. (N.S.) 4, 77–94 (1943). [MF 10354]

The author investigates the motion of an ellipsoid in a uniform stream of an incompressible viscous fluid under three different conditions (neglecting the inertia terms): (1) it always remains a similar ellipsoid; (2) it always remains a confocal ellipsoid; (3) its volume remains constant. Making use of the gravitational potential of the ellipsoid regarded as homogeneous and of unit density, the pressure at the surface of the ellipsoid is obtained in each of the three cases and, from the surface integrals of the force per unit area obtained, the author establishes that the component couples on the ellipsoid are zero. The author concludes the paper by calculating the rate of the dissipation of the energy. *A. Gelbart* (Syracuse, N. Y.).

Dean, W. R. On the shearing motion of fluid past a projection. Proc. Cambridge Philos. Soc. 40, 19–36 (1944). [MF 10799]

In two dimensional slow viscous fluid motion which takes account of the inertial terms, the stream function ψ satisfies

(1) $$\nu\nabla^4\psi=\psi_x\nabla^2\psi_y-\psi_y\nabla^2\psi_x,$$

where ν is the kinematic viscosity coefficient and subscripts denote partial derivatives. The author considers shear flow in the semi-infinite region $y>0$ and above a rigid symmetrical boundary which has sharp or rounded projections on the y-axis to obstruct the flow. On this boundary ψ and its normal derivative vanish and in the infinitely distant portions of the plane ψ attains the value y^2. The rigid boundary is such that $z=w(\zeta)$ maps this boundary conformally on the unit circle in the ζ plane, the function $w(\zeta)$ being a relatively simple rational function of ζ. An approximation solution in the form

$$\psi=\psi_1+k\psi_2+\cdots,$$

where $\psi_1=y^2+\chi$, χ a harmonic function, is determined such that ψ_1 is a solution of $\nabla^4\psi_1=0$. The second approximation is the determination of ψ_2 as a solution of (1) with the known values of ψ_1 inserted in the right member. It is found that, when the boundary has a cusp, a closed vortex exists on the downstream side of the projection but no critical velocity is found. For sharply rounded projections a closed vortex exists only if a certain critical velocity V_c is exceeded, which varies as the one fourth power of the radius of curvature. *D. L. Holl* (Ames, Iowa).

Keulegan, Garbis H. Laminar flow at the interface of two liquids. J. Research Nat. Bur. Standards 32, 303–327 (1944). [MF 10819]

As the starting point of an extensive investigation into the hydrodynamic phenomenon at the interface of two liquids, the author treats the problem of laminar interlayer by assuming small kinematical viscosities for the liquids so that Prandtl's boundary layer equations can be used. If the interface is chosen as the x-axis and y-axis points inward to the upper liquid which is assumed to be moving with a velocity U and y'-axis points inward to the lower liquid which is assumed to be at rest, then the functions H, H' are defined in terms of the stream functions ψ, ψ' in the following way:

$$H(\eta)=-\psi/U\delta,\qquad \eta=y/\delta,\qquad \delta=(2\nu x/U)^{\frac12},$$
$$H'(\eta')=-\psi'/U\delta',\quad \eta'=y'/\delta',\quad \delta'=(2\nu'x/U)^{\frac12},$$

where ν, ν' are the kinematical viscosities of the upper and the lower liquid. Without pressure gradient along the interface, the differential equations are

$$L(\eta)=\frac{d^3H}{d\eta^3}+H\frac{d^2H}{d\eta^2}=0,$$

$$L'(\eta')=\frac{d^3H'}{d\eta'^3}+H'\frac{d^2H'}{d\eta'^2}=0.$$

The author solves the problem by an approximation using a sum of polynomials in η, η'. The approximating polynomials are made to satisfy the usual boundary conditions. To determine the constants in the polynomials, the author suggests two methods. One is essentially Galerkin's method, while the other is new and seems to be justified solely on the ground that it yields results similar to those of Galerkin's method. This new method is based upon the following equations:

$$\int_0^1 Ld\alpha=0,\qquad \alpha=\eta/\eta_s;$$

$$\int_0^1 L'd\alpha'=0,\qquad \alpha'=\eta'/\eta_s',$$

$dL/d\eta=0$, $\eta=\eta_s$; $dL/d\eta=0$, $\eta=0$; $dL'/d\eta'=0$, $\eta'=\eta_s'$; $dL'/d\eta'=0$, $\eta'=0$, where η_s is the value of η corresponding to "thickness" of layer in the upper liquid, and η_s' is the value of η' corresponding to "thickness" of layer in the lower liquid. Numerical results for the various characteristic quantities of the flow are given as functions of r defined as $r^2=\nu'\rho'^2/\nu\rho^2$, where ρ, ρ' are the densities of the upper and the lower liquid. *H. S. Tsien.*

Krahn, E. Die Janzen-Rayleighsche zweite Näherung der kompressiblen Strömung um ein beliebiges Profil. Z. Angew. Math. Mech. 23, 33–35 (1943). [MF 10728]

Following Lord Rayleigh, the author writes the velocity potential $\phi(x, y)$ of a plane irrotational flow of a compressible fluid around a given airfoil profile in the form

$$\phi=U[\varphi+M^2\varphi'+M^4\varphi''+\cdots],$$

where U and M denote velocity and Mach number of the undisturbed stream, and φ, φ', φ'', $\cdots$ are functions of x and y. In particular, φ is the velocity potential of the corresponding incompressible flow, and φ' satisfies

$$\Delta\varphi'=\varphi_x^2\varphi_{xx}+2\varphi_x\varphi_y\varphi_{xy}+\varphi_y^2\varphi_{yy}.$$

A particular integral of this equation is given by

$$\varphi'=\varphi_yf(x, y),$$

where

$$f=\int[\varphi_x\varphi_ydx+\tfrac12(\varphi_y^2-\varphi_x^2)dy].$$

[The author does not seem to be aware of the fact that this constitutes part of the particular integral which I. Imai and T. Aihara [Rep. Aeronaut. Research Inst. Tokyo Imp. Univ., no. 199 (1940)] have expressed in a remarkably elegant form.] It is shown how the velocity distribution corresponding to the first two terms of the series for ϕ can be determined along the profile if the conformal mapping of the exterior of the profile onto the exterior of a circle is known. *W. Prager* (Providence, R. I.).

Stewart, H. J. The aerodynamics of a ring airfoil. Quart. Appl. Math. 2, 136–141 (1944). [MF 10806]

The subject treated is the same as that of Dickmann [Ing.-Arch. 11, 36–52 (1940); these Rev. 2, 171], that is,

an axially symmetric circular-ring wing symmetrically placed in a uniform flow. Instead of handling the elliptic integrals, the present author replaces them by their Fourier integrals. For comparison with the two-dimensional case, he calculates approximately the distribution of radial velocity over the chord for the same circulation distribution as the two-dimensional flat plate; the result indicates that the ring wing requires a slight camber to maintain this circulation distribution. No reference is made to the earlier paper, which was considerably more complete.

W. R. Sears (Inglewood, Calif.).

Magnaradze, L. G. On a new integral equation in the airfoil theory. Bull. Acad. Sci. Georgian SSR [Soobščenia Akad. Nauk Gruzinskoi SSR] 3, 503–508 (1942). (Russian. Georgian summary) [MF 10317]

The author considers Prandtl's well-known singular integro-differential equation

$$(1)\qquad \phi(x)-(b(x)/\pi)\int_{-a}^{a}\phi'(y)dy/(x-y)=f(x),\quad -a<x<a,$$

where $\phi(x)$ is the unknown function and the integral is to be taken as a Cauchy principal value. Assuming that $(a^2-x^2)^{1/2}/b(x)$ has a derivative which satisfies a Hölder condition, the author shows that (1) is equivalent to the regular integral equation

$$(2)\qquad \phi(x)+\pi^{-1}\int_{-a}^{a}K(x,t)\phi(t)dt=F(x),$$

where

$$K(x,t)=\int^{x}\frac{1}{t-\eta}\left[\frac{(a^2-\eta^2)^{1/2}}{b(t)}-\frac{(a^2-\eta^2)^{1/2}}{b(\eta)}\right]\cos\left[\tau(x)-\tau(\eta)\right]d\eta,$$

$$\tau(x)=\int^{x}d\xi/b(\xi)$$

and $F(x)$ is a rather complicated expression involving definite and indefinite integrals. If $(a^2-x^2)^{1/2}/b(x)$ is a rational function, (2) may be reduced to a system of linear algebraic equations. The author states that equation (2) was first obtained by Vecoua under the assumption that $(a^2-x^2)^{1/2}/b(x)$ is an entire function.

L. Bers (Providence, R. I.).

★Gebelein, Hans. Turbulenz. Physikalische Statistik und Hydrodynamik. J. W. Edwards, Ann Arbor, Michigan, 1944. viii+177 pp. $4.00.

The original appeared in Berlin, 1935. The publisher was J. Springer.

Görtler, H. Bemerkung zu: Berechnung von Aufgaben der freien Turbulenz auf Grund eines neuen Näherungsansatzes. Z. Angew. Math. Mech. Bd. 22 (1942) S. 244 bis 254. Z. Angew. Math. Mech. 23, 184 (1943). [MF 10176]

Cf. these Rev. 5, 23.

Sauer, Robert. Charakteristikenverfahren für Kugel- und Zylinderwellen reibungsloser Gase. Z. Angew. Math. Mech. 23, 29–32 (1943). [MF 10729]

In a previous paper [Ing.-Arch. 13, 79–89 (1942); these Rev. 4, 260] the author described a graphical method of solving problems concerning plane one-dimensional flows of incompressible fluids. In the present paper this method is extended to one-dimensional problems with cylindrical or spherical symmetry.

W. Prager (Providence, R. I.).

Tesson, Fernand. Contribution à l'étude des ondes de dérangement. C. R. Acad. Sci. Paris 217, 208–210 (1943). [MF 10643]

It is known that, if a two-dimensional flow of supersonic velocity is made to follow a sudden change in direction by a wedge, a discontinuity of velocity occurs. This continuity is called the oblique shock wave. The author shows that, if the difference in flow directions ahead of the wedge and after the wedge is infinitesimal, the shock wave bisects the angle formed by the Mach wave ahead of the wedge and that after the wedge.

H. S. Tsien (Pasadena, Calif.).

Sorokin, V. On the internal friction of liquids and gases, which have a concealed momentum of rotation. Akad. Nauk SSSR. Zhurnal Eksper. Teoret. Fiz. 13, 306–312 (1943). (Russian) [MF 11222]

Nelson-Skorniakov, F. B. Filtration from bilaterally drained channel of rectangular section. C. R. (Doklady) Acad. Sci. URSS (N.S.) 40, 138–140 (1943). [MF 11166]

Poloubarinova-Kochina, P. J. Concerning "direct and reverse" problems in the hydraulics of a petroleum strata. Appl. Math. Mech. [Akad. Nauk SSSR. Prikl. Mat. Mech.] 7, 361–374 (1943). (Russian. English summary) [MF 11238]

Theory of Elasticity

Carrier, G. F. The thermal-stress and body-force problems of the infinite orthotropic solid. Quart. Appl. Math. 2, 31–36 (1944). [MF 10334]

The states of elastic deformation associated with an arbitrary distribution either of temperature or of body forces in a finite region of an infinite orthotropic elastic solid are expressed in each case in terms of three independent displacement potentials. Each of these potentials is represented by an integral containing a Green's function. This function can be readily determined for two dimensional problems and for certain three dimensional problems.

H. W. March (Madison, Wis.).

Kaşkal, Azmi. On isotropic materials with continuous transition from the elastic to the plastic state. Rev. Fac. Sci. Univ. Istanbul (A) 6, 59–61 (1941). (English. Turkish summary) [MF 10815]

W. Prager [Proc. Fifth International Congress of Applied Mechanics, Cambridge, Mass., 1938, p. 234] has given stress-strain relations for an isotropic incompressible material going from the elastic to the plastic state by a continuous transition. The author formulates stress-strain relations for an isotropic compressible material undergoing the same continuous transition. First, the author shows that if the material is incompressible then his relations reduce to those of Prager. Secondly, the author considers the form assumed by his relations under a tensile test. It is shown that the ratio of the increments of lateral contraction and longitudinal extension depends upon the elastic constants and another ratio, that of the tensile stress to that of the yield stress. Finally, it is shown that the first ratio approximates Poisson's law for small values of the latter ratio; and the first ratio approximates to one-half (its value for an incompressible material) as the latter ratio approaches the limiting value one.

N. Coburn (Austin, Tex.).

Coburn, N. A boundary value problem in plane plasticity. J. Math. Phys. Mass. Inst. Tech. 23, 61–68 (1944). [MF 10627]

A perfectly plastic material occupies the half-plane $x>0$, and the stresses $\sigma_x(x,y)$ and $\sigma_{xy}(x,y)$ are assumed known for $x=0$. The problem at hand is the determination of the stresses acting at any point in the half-plane $x>0$. By introducing the Airy stress function, this problem can be reduced to a pair of nonhomogeneous linear hyperbolic differential equations for the Airy stress function. From the solution of this problem, the author formulates a mathematical condition for the determination of the "boundary curve beyond which the statically determinate plastic stress distribution cannot be extended." *A. E. Heins* (Cambridge, Mass.).

Alfrey, T. Non-homogeneous stresses in visco-elastic media. Quart. Appl. Math. 2, 113–119 (1944). [MF 10804]

The author develops a theory for small strains in an isotropic, incompressible, visco-elastic material. First, he considers an elastic incompressible material. By use of the stress-strain relations, the incompressibility condition and the equilibrium relations (neglecting the inertia term), it is shown that, in the elastic case, the stresses satisfy

$$(1) \qquad \sigma_{ik,ll}+2\sigma_{,ik}=0, \qquad i,k,l=1,2,3.$$

Subscripts following the comma indicate partial differentiation; the summation convention is used for repeated indices; σ_{ik} denote the stress components and σ the mean normal stress. It is assumed that these equations are sufficient to determine the stresses. Next, generalizing the Voigt and Maxwell stress-strain relations, the author considers a visco-elastic material with stress-strain relations of the type

$$(2) \qquad Ps_{ik}=Q\epsilon_{ik},$$

where

$$(3) \qquad P=\frac{\partial^m}{\partial t^m}+a_{m-1}\frac{\partial^{m-1}}{\partial t^{m-1}}+\cdots+a_0, \qquad Q=b_n\frac{\partial}{\partial t^n}+b_{n-1}\frac{\partial^{n-1}}{\partial t^{n-1}}+\cdots+b_0;$$

s_{ik} is the stress-deviator $(\sigma_{ik}-\sigma\delta_{ik})$; ϵ_{ik} is the strain tensor. By proceeding as in the elastic case, the author finds that the stresses must satisfy

$$(4) \qquad P(\sigma_{ik,ll}+2\sigma_{,ik})=0.$$

Assuming that the equations (4) are sufficient to determine the stresses and that these stresses (and the surface forces) are analytic functions of time, the author concludes: in the case of the first boundary value problem, the stress distribution in an incompressible visco-elastic material is identical with that in an incompressible elastic material under the same instantaneous surface forces. As an illustration of this result, the author considers the case where the space and time variables are separable in the surface forces. It follows that the stresses and strains are separable and the visco-elastic relations reduce to ordinary differential equations in time for the determination of the response function. Finally, using an integral representation for the surface forces involving Heaviside's unit function (and hence, as the author states, nonanalytic and contradictory to the original assumptions), the author states without proof that the stresses and strains can be obtained by an integral representation involving the response function. As an example, the author considers a thin cantilever beam subjected to transverse loading and consisting of a Voigt material. [Reviewer's note: in view of the author's neglect of the inertia term, it appears that he is considering a material whose density is negligible in comparison to a_j, b_j in some sense. This point should be clarified.]

N. Coburn (Austin, Tex.).

Dean, W. R., Parsons, H. W. and Sneddon, I. N. A type of stress distribution on the surface of a semi-infinite elastic solid. Proc. Cambridge Philos. Soc. 40, 5–19 (1944). [MF 10798]

The authors consider the problem of determining the conditions at the surface of a semi-infinite elastic body which has an interior force whose direction is at right angles to the surface. For a single interior point load the problem was solved by Mindlin [Physics 7, 195–202 (1936)]. These results are again obtained by the authors by employing two equal and oppositely directed loads applied at points which are image points with respect to the surface and superposing a suitable stress system to neutralize the stresses induced at the boundary surface. The principle of superposition is utilized to solve the surface effects arising when the internal loading is a uniformly distributed tension over a finite circular area at a finite depth from the surface. The analysis is carried out for an incompressible medium (Poisson's ratio equals 0.5) and the results involve the normal elliptic integrals of the first and second kind. A comparison is made with the results of the same loading on the surface previously solved by Love. *D. L. Holl* (Ames, Iowa).

***Sokolovsky, W. W. Statics of Earthy Mediums.** Akademia Nauk SSSR, 1942. 207 pp. (Russian. English chapter summaries)

The book is concerned with the mathematical theory of statically determinate distributions of plane stress in an earthy medium which has attained a limit state of equilibrium. The stress components σ_x, σ_y and τ satisfy the two conditions of equilibrium and the condition characterizing limit state of equilibrium, namely,

$$(1) \qquad (\sigma_x-\sigma_y)^2+4\tau^2=(\sigma_x+\sigma_y+2k\cot\rho)^2\sin^2\rho,$$

where the coefficient of cohesion k and the angle of friction ρ are constants. The condition (1) is satisfied by setting

$$\sigma_x=\sigma[1+\sin\rho\sin(2\varphi+\rho)]-k\cot\rho,$$
$$\sigma_y=\sigma[1-\sin\rho\sin(2\varphi+\rho)]-k\cot\rho,$$
$$\tau=-\sigma\sin\rho\cos(2\varphi+\rho).$$

The equilibrium conditions

$$\frac{\partial\sigma_x}{\partial x}+\frac{\partial\tau}{\partial y}=X, \quad \frac{\partial\tau}{\partial x}+\frac{\partial\sigma_y}{\partial y}=Y$$

lead then to differential equations for σ and φ, which are of hyperbolic type. Important types of boundary value problems are formulated, and the questions of existence and uniqueness of the solution are discussed. Following ideas of Massau, a procedure of constructing approximate solution is developed. The discussion is extended to include the case of a nonhomogeneous medium for which k and ρ are given functions of x and y, and the case of an anisotropic medium for which the values of k and ρ corresponding to the horizontal plane are smaller than the values corresponding to any other plane. The following problems are discussed in detail: Prandtl's problem (the half plane $x\geqq 0$ is exposed to given stresses σ_x and τ along the positive y-axis; to determine the smallest or greatest values of the normal stresses

σ_x along the negative y-axis which will ensure equilibrium); determination of the pressures on sustaining walls of various shapes; stability of slopes. An appendix contains several examples worked out completely. *W. Prager.*

Kapuano, I. Sur une nouvelle propriété des réseaux de Hencky-Prandtl. Rev. Fac. Sci. Univ. Istanbul (A) 6, 36–39 (1941). (French. Turkish summary) [MF 10813]

The author considers a perfectly plastic plane material (yielding under constant maximum shearing stress). It is known that, for such a material, the slip lines coincide with the lines of maximum shearing stress. H. Hencky [Z. Angew. Math. Mech. 3, 241–251 (1924)], L. Prandtl [Z. Angew. Math. Mech. 3, 401–406 (1923)] and W. Prager [Rev. Fac. Sci. Univ. Istanbul (A) (N.S.) 4, 22–24 (1938)] have determined characteristic properties of this net of slip lines (called a Hencky-Prandtl net). The author of this article demonstrates a further property of this net: the evolutes of the curves of one family of a Hencky-Prandtl net form one family of a new Hencky-Prandtl net; the orthogonal trajectories of these evolutes are the involutes of the other family of the original net. *N. Coburn* (Austin, Tex.).

Shapiro, G. S. The bending of semi-infinite plates rested on the elastic foundation. Appl. Math. Mech. [Akad. Nauk SSSR. Prikl. Mat. Mech.] 7, 316–320 (1943). (Russian. English summary) [MF 11233]

The author gives the results of computations: (a) for the plate loaded by a concentrated force on the edge; (b) for a uniformly distributed load within a range of the edge; (c) for a uniformly distributed load along a range of a straight perpendicular to the edge of the plate.

Author's summary.

Vekua, Ilja. On the bending of a plate with free boundaries. Bull. Acad. Sci. Georgian SSR [Soobščenia Akad. Nauk Gruzinskoi SSR] 3, 641–648 (1942). (Russian. Georgian summary) [MF 10328]

Let the middle surface of a thin elastic plate occupy a domain T of the (x, y)-plane and let T be bounded by a smooth simple closed curve C. Under action of a force $p(x, y)$ applied orthogonally to the center surface, this surface is distorted into a surface given by $z=z(x, y)$. Boundary conditions corresponding to a clamped edge, a simply supported edge, and a free edge, as well as mixed boundary conditions, are considered. For the free edge boundary condition, the deflection function $z=z(x, y)$ is determined by showing the problem to be equivalent to a fundamental problem of plane elasticity of which the solution is known. Acknowledgment is made of the previous use of the same method by S. G. Lekhnitsky [Appl. Math. Mech. [Akad. Nauk SSSR. Prikl. Mat. Mech.] (N.S.) 2, 181–209 (1938)]. *E. F. Beckenbach* (Austin, Tex.).

Federhofer, Karl. Berechnung der dünnen Kreisplatte mit grosser Ausbiegung. Luftfahrtforschung 21, 1–10 (1944). [MF 10882]

Holgate, S. The effect of a hole on certain stress distributions in aeolotropic and isotropic plates. Proc. Cambridge Philos. Soc. 40, 172–188 (1944). [MF 10786]

The following problems are solved by expressing the stress components as functions of complex variables: equal and opposite radial forces at the extremities of a diameter having an arbitrary orientation with respect to the axes of elastic symmetry; two equal and opposite forces at opposite ends of a chord parallel to an axis of symmetry; isolated force at any point on the boundary; force at a point in the material near the edge of the hole and parallel to either of the coordinate axes in the case of isotropic material and parallel to either of the axes of elastic symmetry in the case of aelotropic material; stresses along straight boundaries as limiting cases. For the method employing functions of complex variables reference is made to several papers by A. E. Green [for one of them see Proc. Roy. Soc. London. Ser. A. 180, 173–208 (1942); these Rev. 4, 123] and to one of several papers by S. G. Lechnitzky [C. R. (Doklady) Acad. Sci. URSS (N.S.) 4, 111–115 (1936)]. For the use of functions of complex variables in problems connected with isotropic materials reference might properly be made to E. Goursat [Bull. Soc. Math. France 26, 236–237 (1898)] and to N. I. Muskhelišvili [Bull. Acad. Sci. URSS (6) 13, 663–686 (1919); Z. Angew Math. Mech. 13, 264–282 (1933)]. *H. W. March* (Madison, Wis.).

Muschelišvili, N. I. The fundamental boundary value problems of the theory of elasticity for a half-plane. Bull. Acad. Sci. Georgian SSR [Soobščenia Akad. Nauk Gruzinskoi SSR] 2, 873–880 (1941). (Russian. Georgian summary) [MF 10307]

This note deals with the state of plane stress in an elastic half-plane ($y<0$). It is assumed that the boundary is stress free except for a finite number of segments $L_k=(a_k, b_k)$, that the stresses vanish at infinity and that the vector sum of the external forces is finite. The author shows that the stress components X_x, Y_y, X_y and the displacements u, v may be written in the form:

$$X_x+Y_y=2[\phi(z)+\overline{\phi(z)}],$$
$$Y_y-iX_y=\phi(z)-\phi(\bar{z})+(z-\bar{z})\bar{\phi}'(\bar{z}),$$
$$2\mu(u'+iv')=\kappa\phi(z)+\phi(\bar{z})-(z-\bar{z})\bar{\phi}'(\bar{z}).$$

Here $\phi(z)$ is an analytic function of $z=x+iy$, regular in the whole plane except the slits L_k; $u'=\partial u/\partial x$, $v'=\partial v/\partial x$ and μ, κ are constants. These formulas are used for simple and effective computation of the stress distribution under either of the following boundary conditions: (i) Y_y and X_y are given along a segment of the real axis; the remaining boundary is stress free; (ii) u and v are given along the whole real axis; (iii) u and v are given along n segments L_k; (iv) v is given along n segments L_k and $X_y=0$ along $0x$. In the last two problems the resultants of the external forces are given. *L. Bers* (Providence, R. I.).

Muschelišvili, N. I. The fundamental boundary value problems of the theory of elasticity for a plane with rectilinear slits. Bull. Acad. Sci. Georgian SSR [Soobščenia Akad. Nauk Gruzinskoi SSR] 3, 103–110 (1942). (Russian. Georgian summary) [MF 10324]

The author determines the plane stress distribution in an elastic plane with a finite number of slits along the real axis. He assumes that either the displacements or the stresses are given along the slits. The method is similar to the one described in the note reviewed above. It is based on a representation of the stresses and the displacements by means of two analytic functions of a complex variable. The author also outlines the application of his method to a "mixed" boundary value problem previously treated by D. I. Schermann [C. R. (Doklady) Acad. Sci. URSS (N.S.) 27, 329–333 (1940); these Rev. 2, 270]. In this problem the stresses are given along the upper sides of the slits and the displacements along the lower sides. *L. Bers.*

AUTHOR INDEX

PRINTED IN THE UNITED STATES OF AMERICA